PHYS 506 Test 1

February 2017

***Asterisked sections are for graduate students only.**

1. [10 points]

   An Einstein solid has 4 oscillators and 4 units of energy. What is the multiplicity of states? $\Omega = \binom{N+q-1}{q} = \binom{7}{4} = \frac{7!}{4!\,3!} = 35.$

2. [10 points]

   Consider N molecules of an ideal monatomic gas, each of mass m, and confined to a volume V. If the internal energy of the gas is U, the entropy is given by the "Sackur-Tetrode" equation

   $T \equiv \left(\frac{\partial S}{\partial U}\right)_{V,N}^{-1}$

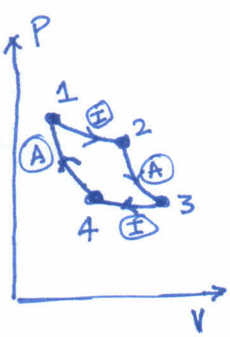
   $$S \approx Nk\left\{\ln\left(\frac{V}{N}\left(\frac{4\pi mU}{3Nh^2}\right)^{3/2}\right) + \frac{5}{2}\right\}$$

   $T = \left[Nk \frac{\partial(\ln U^{3/2})}{\partial U}\right]^{-1} = \frac{2}{3}\frac{U}{Nk}.$

   $\Rightarrow U = \frac{3}{2}NkT$

   Use this equation to find the temperature of the gas, and relate that to the internal energy. How does this result compare with what you expect from the equipartition of energy? This is what equipartition of energy predicts: $U = (f/2)NkT$, where $f=3$ for a monatomic gas.

3. [10 points]



   Undergraduates:
   Define the Carnot cycle for an ideal gas engine on a PV-diagram and, for each step, find the change in internal energy of the gas ΔU, the heat added to the gas Q, and the work done on the gas W. You may assume that the quantity PV^γ is a constant along an adiabat, where $\gamma > 1$.

   Isotherms: I
   Adiabats: A
   For isotherms, $\Delta U = 0$.
   For adiabats, $Q = 0$.

   ***Graduate Students:** $\Delta U = Q + W.$
   Find the efficiency of a Carnot engine operating between temperatures T_H and T_C and show that this is the most efficient reversible engine that can operate using heat baths at the two given temperatures. $W_I = -\int P\,dV = -\int \frac{nRT}{V}\,dV = -nRT\ln(V_f/V_i)$

Adiabats: $PV^\gamma = \text{const.} \Rightarrow P\gamma V^{\gamma-1}dV + V^\gamma dP = 0.$

$\Rightarrow P\gamma\,dV = -V\,dP. \Rightarrow dU = \frac{3}{2}d(PV) = \frac{3}{2}(P\,dV + V\,dP) = \frac{3}{2}(1-\gamma)P\,dV.$

$\Rightarrow W = -\int P\,dV = -\int \frac{d(PV)}{(1-\gamma)} = \frac{-\Delta(PV)}{(1-\gamma)}.$

Step	ΔQ	ΔW	ΔU
$1\to2$	$nRT_H\ln\left(\frac{V_2}{V_1}\right)$	$-\Delta Q$	0
$2\to3$	0	$-\Delta(PV)/(1-\gamma)$	ΔW
$3\to4$	$nRT_c\ln\left(\frac{V_4}{V_3}\right)$	$-\Delta Q$	0
$4\to1$	0	$-\Delta(PV)/(1-\gamma)$	ΔW
Total	$nRT_H\ln\left(\frac{V_2}{V_1}\right) + nRT_c\ln\left(\frac{V_4}{V_3}\right) = nR(T_H - T_c)\ln(V_2/V_1)$.	$-\Delta Q_{tot}$	0

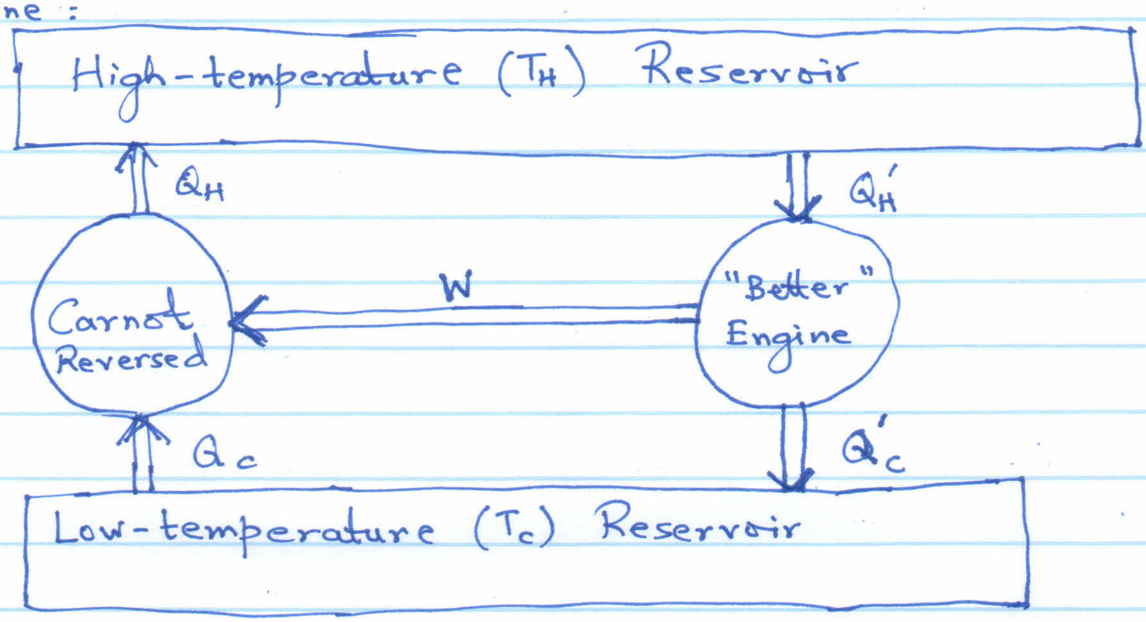
For adiabats, $P_2V_2^\gamma = P_3V_3^\gamma$, $P_1V_1^\gamma = P_4V_4^\gamma$.

Thus, $\left(\frac{P_2}{P_1}\right)\left(\frac{V_2}{V_1}\right)^\gamma = \left(\frac{P_3}{P_4}\right)\left(\frac{V_3}{V_4}\right)^\gamma.$

Using $(P_2/P_1) = (V_1/V_2)$ for an isotherm, and similarly $(P_3/P_4) = (V_4/V_3)$, we get $\frac{V_2}{V_1} = \frac{V_3}{V_4}.$

$$\epsilon = \frac{-\Delta W_{tot}}{Q_H} = \frac{nR(T_H - T_c)\ln(V_2/V_1)}{nRT_H\ln(V_2/V_1)} = \frac{T_H - T_c}{T_H}.$$

*3 b) Run the Carnot engine in reverse and see what happens to the total entropy when connected to a "better" engine:



Since the other engine is "better", $Q_H' < Q_H$ to produce the same work W. Also, $Q_c' = Q_H' - W$ is thus less than Q_c.

Let $\Delta Q = Q_H - Q_{H'} = (Q_c + W) - (Q_{c'} + W) = (Q_c - Q_{c'})$.

The net effect of the two engines is to extract positive heat ΔQ from the T_c reservoir and send it to the T_H reservoir. Thus,

$$\Delta S_{total} = \Delta S_c + \Delta S_H = -\frac{\Delta Q}{T_c} + \frac{\Delta Q}{T_H}.$$

Since $T_H > T_c$, $\Delta S_{total} < 0$, which is not allowed by the II Law.