7.46) (a) Using (7.86) and (7.89) we get

$$F = U - TS = \frac{8\pi^5 (kT)^4}{15 (hc)^3} V - \frac{32\pi^5}{45} \frac{(kT)^4}{(hc)^3} V$$

$$= -\frac{8}{45} \pi^5 \frac{(kT)^4}{(hc)^3} V = -\frac{U}{3}.$$

(b) $$-\partial_T F = \frac{1}{3}(\partial_T U) = \frac{32\pi^5 (kT)^3 . k V}{45 (hc)^3} = S.$$

(c) $$P = -\partial_V F = \frac{1}{3}(\partial_V U) = \frac{8}{45} \frac{\pi^5 (kT)^4}{(hc)^3} = \frac{1}{3}\frac{U}{V},$$ in agreement with problem (7.45).

(d) For a single mode, $F = -kT \ln Z = -kT \ln\left(e^{-\hbar\omega/2kT} + e^{-3\hbar\omega/2kT} + \dots\right)$, i.e.,

$$F = -kT \ln\left[\frac{e^{-\hbar\omega/2kT}}{(1 - e^{-\hbar\omega/kT})}\right].$$ Summing over all modes we get

$$F = \iiint dn_x\, dn_y\, dn_z \quad (+kT)\, e^{-\hbar\omega_n/2kT} \quad \ln\left(1 - e^{-\hbar\omega_n/kT}\right)$$

Using $\omega_n = 2\pi\nu = 2\pi \cdot \frac{nc}{2L}$, and using spherical coordinates so $dn_x\, dn_y\, dn_z \to n^2 dn\, d\Omega_n$, integrating over the positive octant (so that $\int d\Omega_n = \frac{4\pi}{8}$) and multiplying by 2 for the two photon polarizations, we get (ignoring the ground state prefactor)

$$F = 2kT \cdot \frac{\pi}{2} \cdot \int_0^\infty n^2 dn \ \ln\left(1 - e^{-n(\hbar\pi c/L)/kT}\right)$$

Setting $n \frac{\hbar\pi c}{LkT} = x$, we get

$$F = \pi kT \cdot \left(\frac{LkT}{\hbar\pi c}\right)^3 \int_0^\infty x^2 dx \ \ln(1 - e^{-x}) = \frac{\pi (kT)^4 (2L)^3}{(hc)^3} \int_0^\infty x^2 dx \ln(\dots)$$

Now $$\int_0^\infty x^2 dx \ln(1 - e^{-x}) = \ln(1 - e^{-x}) \frac{x^3}{3}\Big|_0^\infty - \int_0^\infty \frac{dx\, x^3}{3(1 - e^{-x})} = -\frac{1}{3} \cdot \frac{\pi^4}{15}.$$

Thus, $$F = -\frac{1}{3} \cdot \frac{8\pi^5 (kT)^4 V}{15 (hc)^3} = -\frac{U}{3}.$$