1. Prove that \( \frac{d}{dx} J_{n}(x) = - \frac{J_{n+1}(x) - J_{n-1}(x)}{x} \).

2. Many facts about Bessel functions can be proved by using its generating function
\[ g(x, t) = e^{\frac{x(t-1)}{2}}. \] Prove that
\[ e^{\frac{x(t-1)}{2}} = \sum_{n=-\infty}^{\infty} J_{n}(x)t^{n}. \]

3. From the product of generating functions \( g(x, t) \cdot g(x, -t) \) show that
\[ 1 = [J_{0}(x)]^{2} + 2[J_{1}(x)]^{2} + 2[J_{2}(x)]^{2} + \ldots \]

4. Use the generating function to show that \( J_{n}(x) = (-1)^{n} J_{n}(-x) \).

5. Use the generating function to show that \( 2J_{n}'(x) = J_{n-1}(x) - J_{n+1}(x) \).

6. Using mathematical induction derive \( J_{n}(x) = (-1)^{n} x^{n} \left( \frac{1}{x} \frac{d}{dx} \right)^{n} J_{0}(x) \).