Ratio of the proton electromagnetic form factors from meson dressing

S. Kondratyuk
Department of Physics and Astronomy, University of South Carolina, Columbia, South Carolina 29208, USA and
Department of Physics and Astronomy, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2

K. Kubodera and F. Myhrer
Department of Physics and Astronomy, University of South Carolina, Columbia, South Carolina 29208, USA
(Received 10 November 2004; published 24 February 2005)

The dressed $K$-matrix model, developed previously for low- and intermediate-energy Compton scattering, is
generalized to calculate the photon-nucleon vertex with a virtual photon and an off-shell nucleon. This model
was used to compute the ratio of the proton electromagnetic form factors. The calculated ratio is in excellent
agreement with the ratio measured in polarized electron-proton scattering.

DOI: 10.1103/PhysRevC.71.028201 PACS number(s): 13.40.Hq, 12.40.−y, 11.55.−m, 25.30.Bf

I. INTRODUCTION

Results of the recent experiments [1] on polarized electron-proton scattering and their apparent disagreement [2] with
the electromagnetic form factors (EM FFs) extracted from Rosenbluth cross-section measurements indicate that much
remains to be learned about nucleon EM interactions. While a definitive understanding of the nucleon EM FFs is still lacking,
several theoretical models [3–8] are able to describe their momentum-transfer dependence consistently with the JLab
experiments [1].

Since the ultimate goal is to reach a coherent understanding of all aspects of the nucleon EM interactions, important
insights can be gained by describing as many different photon-nucleon reactions as possible in a unified theoretical
approach. The dressed $K$-matrix model (DKM) [9–11] has been developed for a description of real Compton scattering in
a wide energy region both below and above the pion production threshold. The central element of this model is a nonperturba-
tive nucleon dressing based on the iterative use of dispersion relations, thus adding analyticity constraints to the properties
of relativistic covariance, unitarity, crossing symmetry, and gauge invariance of the usual $K$-matrix approaches.

The parameters of the model are not entirely free: the convergence requirement of the dressing procedure imposes
constraints on the allowed range of these parameters while introducing an interdependence among them. The thus-
constrained parameters were then completely fixed by a fit to Compton cross sections and pion-nucleon ($\pi N$) phase shifts at
intermediate energies [10]. With all the parameters fully fixed, low-energy observables—such as the nucleon polarizabilities
[10], pion-nucleon scattering lengths, and $\Sigma$ term [11]—were then calculated and shown to agree with experimental data.
The main reason for this success is that important analyticity (causality) constraints are implemented in DKM by dressing the $\gamma NN$, $\pi NN$, and $\pi N\Delta$ vertices and the nucleon and $\Delta$ propagators with meson loops up to infinite order. The extent to
which analyticity is fulfilled can be quantified by comparing the low-energy amplitudes with corresponding sum rules, both evaluated within the same model. Within DKM, such a comparison was made for the Baldin-Lapidus and

Gerasimov-Drell-Hearn sum rules vs. the nucleon polarizabilities [12] and for the Adler-Weisberger and Goldberger-Miyazawa-Oehme sum rules vs. the near-threshold $\pi N$ amplitude [13]. We generally found good agreement between the low-energy and sum-rule evaluations.

In this report we extend the dressing procedure to the case of virtual spacelike photons. The resulting dressed photon-
nucleon ($\gamma NN$) vertex has six invariant functions depending on two invariant variables: the four-momentum squared of the
photon and of one of the nucleons, the other nucleon being on-shell. By putting both nucleons on the mass shell we calculate
the momentum-transfer dependence of the ratio of the proton EM FFs. We find good agreement of our calculation with the
JLab polarization-transfer measurements [1].

The crucial point of this result is that the momentum-transfer dependence of the ratio of the FFs is calculated here with
the same parameters as used in Refs. [10–13] to describe the Compton and $\pi N$ amplitudes at low and intermediate
energies. This shows that the dynamics of DKM captures features which are important in a wide kinematical range
relevant to the photon-nucleon interactions.

II. FORM FACTORS IN THE DRESSED $\gamma NN$ VERTEX

The most general Lorentz-covariant structure of the $\gamma NN$ vertex with an incoming off-shell photon (four-momentum $p$) and
an incoming off-shell nucleon (four-momentum $q$) can be written as [14]

\[ \Gamma^{\mu}(p,q) = \sum_{r=\pm} \left( \gamma^{\mu} F^r(p^2, q^2) + i \frac{\sigma^{\mu\nu} q^\nu}{2m} F^r(p^2, q^2) \right) \Lambda_r(q), \]

where $\Lambda_\pm(q) = (\pm q + m)/(2m)$ are the nucleon positive- and negative-energy projection operators,\(^1\) and the six invariant

\(^1\)Throughout the paper we use the conventions and definitions of Ref. [15].
functions $F^i_r(p^2, q^2) (i = \{1, 2, 3\}, r = \pm)$ are called half-off-shell FFs. The outgoing nucleon is on the mass shell, i.e., its four-momentum $p' = p + q$ obeys $p'^2 = m^2$. We consider protons interacting with real or virtual spacelike photons, i.e., $q^2 \leq 0$.

The usual Dirac and Pauli FFs, denoted as $F_D(q^2)$ and $F_P(q^2)$, respectively, are obtained by putting both nucleons on the mass shell in Eq. (1):

$$F_D, P(q^2) = \lim_{p^2 \to m^2} F_{1, 2}^+ (p^2, q^2). \tag{2}$$

When the two nucleons are on-shell, $F_{1, 2}^+(m^2, q^2)$ do not enter the vertex since $\Delta_-(q^2 = m) = 0$; in addition, space-time reflection invariance requires [14] that

$$F_3^+(m^2, q^2) = 0. \tag{3}$$

$F_{D, P}(q^2)$ are related to the Sachs (electric and magnetic) FFs $G_{E, M}(q^2)$ by

$$G_E(q^2) = F_D(q^2) + \frac{q^2}{4m^2} F_P(q^2), \quad G_M(q^2) = F_D(q^2) - F_P(q^2). \tag{4}$$

The $\gamma N N$ vertex Eq. (1) is calculated by dressing a bare vertex with an infinite number of meson loops including pions and vector isoscalar mesons. The $\pi N N$ and $\pi N A$ vertices and the nucleon and $\Delta$ propagators, which enter in the loop integrals for the $\gamma N N$ vertex, are also dressed nonperturbatively with pions and $\rho$ and $\sigma$ mesons. The detailed description of the dressing technique for the $\gamma N N$ vertex (with real photons) can be found in Ref. [9], and for the $\pi N N$, $\pi N A$ vertices and the propagators in Ref. [11]. This dressing is part of DKM [10] where it amounts to restoring the principal-value parts of loop contributions to the Compton and $\pi N$ amplitudes.

In the present work we have generalized the dressing procedure of Ref. [9] from real to virtual spacelike photons in Eq. (1). Thus we have calculated the six invariant functions $F^i_r(p^2, q^2)$ for $-\infty < p^2 < \infty$ and $q^2 \leq 0$, from which the momentum-transfer dependence of the EM FFs has been obtained according to Eqs. (2) and (4). For brevity, we will focus only on the important new features related to the off-shellness of the photon.

We write the bare vertex as

$$\gamma^\mu + i \frac{\sigma^{\mu\nu} q^\nu}{2m} \kappa B + \frac{q^\mu}{m} h_B(q^2). \tag{5}$$

The renormalization consists in choosing the bare constant $\kappa_B$ such that the dressed vertex with all particles on-shell reproduces the physical anomalous magnetic moment of the proton, i.e., $F_2^+(m^2, 0) = \kappa_B = \mu_p - 1 = 1.79$, and adjusting $h_B(q^2)$ so that the calculated function $F_3^+(p^2, q^2)$ obeys Eq. (3).

### III. GAUGE INVARIANCE OF THE MODEL

The Ward-Takahashi identity (WTI) is a consequence of gauge invariance of the theory [16]. It relates the $\gamma N N$ vertex Eq. (1) to the propagator $S(q^2)$ of the off-shell proton:

$$q \mu \Gamma^\mu (p, q) = -S^{-1}(q).$$

Thus, the WTI dictates that the half-off-shell FFs in Eq. (1) must obey the relations

$$F_3^+(p^2, q^2) = \frac{m}{p^2 - m^2} \left( \frac{\alpha(p^2)}{m} \frac{(p^2 + m^2)}{m} - 2\alpha(p^2)\xi(p^2) \right. \tag{6}$$

$$+ q^2 \frac{1}{2m^2} (3m^2 + p^2) F_3^+(p^2, q^2) \left. + (m^2 - p^2) F_3^-(p^2, q^2) \right),$$

$$F_3^-(p^2, q^2) = \alpha(p^2) + \frac{q^2}{2m^2} \left[ F_3^+(p^2, q^2) - F_3^-(p^2, q^2) \right], \tag{7}$$

where the self-energy functions $\alpha(p^2)$ and $\xi(p^2)$ enter into the dressed nucleon propagator $S(q) = [\alpha(p^2)(p^2 - \xi(p^2))]^{-1}$ which was calculated together with the dressed $\Delta$ propagator and $\pi N N$ and $\pi N A$ vertices [11]. The renormalization conditions $S^{-1}(m) = 0$ and Res $S(m) = 1$ can be explicitly written as

$$\alpha(m^2) (m - \xi(m^2)) = 0, \tag{8}$$

$$2m(m - \xi(m^2)) \frac{d\alpha(p^2)}{dp^2} \bigg|_{p^2 = m^2} - \alpha(m^2) \left( \frac{2m}{p^2 - m^2} \left[ \frac{d\xi(p^2)}{dp^2} \right]_{p^2 = m^2} - 1 \right) = 1. \tag{9}$$

On expanding Eq. (6) in powers of $p^2$ around $m^2$ and using Eqs. (2), (3), (8), and (9), we obtain the WTI for the Dirac FF of the proton as

$$F_D(q^2) = 1 + q^2 \left( \frac{\partial F_3^+(p^2, q^2)}{\partial p^2} \bigg|_{p^2 = m^2} - F_3^+(m^2, q^2) \right) \tag{10}$$

yielding the familiar constraint $F_D(q^2 = 0) = 1$ for an on-shell vertex with a real photon. Note that we do not impose Eqs. (6) and (7) “by hand” at any stage of the dressing; nevertheless, the resulting vertex does obey these constraints of the WTI because of the gauge invariance of the model.

### IV. RESULTS OF THE CALCULATION

The ratio of the calculated proton EM FFs is shown in Fig. 1 as a function of the photon momentum squared, $Q^2 \equiv -q^2 > 0$, together with the recent JLab measurements [1].

We found that in order to obtain accurate agreement with experiment over the wide range of momenta-transferred $Q^2$, the coupling of the vector isoscalar particle to the nucleon has to be different from the $\omega N N$ coupling given in Ref. [10]. While the part of the vertex proportional to $\gamma_\mu$ is comparable to that of the $\omega N N$ vertex (instead of $g_\omega = 12$ as in [10], now we use $g_{\omega \text{ vect}} = 8$), the part proportional to $\sigma_\mu q^\nu$ has to be quite
large (we need $\kappa_{\text{vect}} = 60$ instead of $\kappa = -0.8$). The other coupling constants are the same as in [10], i.e., as fixed in the description of the $\pi N$ and Compton scattering processes. It is important that even without the contribution of the vector isoscalar particle, we find the slope of the ratio of the FFs to be negative at $Q^2 = 0$, although in that case it is too steep. Including the vector particle in the dressing makes the slope less steep. A negative slope of the ratio implies that the electric polarizability of the proton, $\alpha_E > \beta_M$, is important that even without the contribution of the vector particle, we find the slope of the ratio of the FFs to $\alpha_E > \beta_M$. This is a well-known fact that the electric polarizability of the proton, $\alpha_E$, is larger than its magnetic polarizability, $\beta_M$, $\approx 2 \times 10^{-4}$ fm$^3$, that is, $\alpha_E \approx 12 \times 10^{-4}$ fm$^3$, is larger than its magnetic polarizability, $\beta_M \approx 2 \times 10^{-4}$ fm$^3$, that is, $\alpha_E > \beta_M$. Thus, one might speculate that the proton behaves as a “larger” and hence more “deformable” object in the presence of an electric field than in the presence of a magnetic field. Since

\[ \frac{d}{dQ^2} \frac{G_E(Q^2)}{G_M(Q^2)} \bigg|_{Q^2=0} < 0, \tag{11} \]

then taking into account that $G_E(Q^2 = 0) = 1$, $G_M(Q^2 = 0) = \mu_p$, and using the usual definitions of the mean-square radii

\[ \langle r_{E,M}^2 \rangle = -\frac{d}{dQ^2} \frac{G_{E,M}(Q^2)}{Q^2} \bigg|_{Q^2=0}, \tag{12} \]

we find

\[ \langle r_{E}^2 \rangle > \frac{\langle r_{M}^2 \rangle}{\mu_p}. \tag{13} \]

Note that the traditional one-photon exchange analyses of the Rosenbluth cross sections would yield $\mu_p G_E(Q^2) / G_M(Q^2) = \text{const} \approx 1$ (see, e.g., Ref. [2]), which would imply equal electric and (normalized) magnetic radii, in sharp contrast to Eq. (13). We would also like to point out that the relation in Eq. (13) is consistent with the well-known fact that the electric polarizability of the proton, $\alpha_E \approx 12 \times 10^{-4}$ fm$^3$, is larger than its magnetic polarizability, $\beta_M \approx 2 \times 10^{-4}$ fm$^3$, that is, $\alpha_E > \beta_M$. Thus, one might speculate that the proton behaves as a “larger” and hence more “deformable” object in the presence of an electric field than in the presence of a magnetic field.

$$\mu_p G_E(Q^2) / G_M(Q^2) = \text{const} \approx 1$$ (see, e.g., Ref. [2]), which would imply equal electric and (normalized) magnetic radii, in sharp contrast to Eq. (13). We would also like to point out that the relation in Eq. (13) is consistent with the well-known fact that the electric polarizability of the proton, $\alpha_E \approx 12 \times 10^{-4}$ fm$^3$, is larger than its magnetic polarizability, $\beta_M \approx 2 \times 10^{-4}$ fm$^3$, that is, $\alpha_E > \beta_M$. Thus, one might speculate that the proton behaves as a “larger” and hence more “deformable” object in the presence of an electric field than in the presence of a magnetic field. Since the mean-square radii are extracted from the momentum-transfer dependence of the FFs while the polarizabilities come from Compton scattering, the dressing procedure of DKM could provide a theoretical framework for understanding this (possibly fortuitous) similarity between Eqs. (13) and (14).

Even though the ratio of the calculated FFs agrees with experiment very well as shown in Fig. 1, the model fails in describing $G_E(Q^2)$ and $G_M(Q^2)$ separately. This is demonstrated in Fig. 2 for the electric FF. A possible remedy could be to use an appropriate FF in the bare $\gamma NN$ vertex, which would introduce additional cutoff parameters. We have not done so in the present calculation since our aim is to keep the same model parameters as fixed in describing Compton and $\pi N$ scattering. In this way we impose a stringent dynamical constraint on the model.

The dependence of the calculated $\gamma NN$ vertex on the momentum squared $p^2$ of the off-shell proton is illustrated in Fig. 3, where we show the $p^2$ dependence of the ratio of generalized (half-off-shell) Sachs FFs, which we define as

\[ G_{E,\text{gen}}^E(p^2, q^2) = F_{1E}^E(p^2, q^2) + \frac{q^2}{4m^2} F_{2E}^E(p^2, q^2), \]

\[ G_{M,\text{gen}}^E(p^2, q^2) = F_{1E}^M(p^2, q^2) + F_{2E}^M(p^2, q^2). \tag{15} \]

The usual Sachs FFs (4) are obtained from (15) by taking the limit $p^2 \to m^2$ in accordance with Eq. (2).

**FIG. 1.** Momentum-transfer dependence of the ratio of the proton EM FFs Eq. (4) ($Q^2 = -q^2$). The present calculation is compared with the experimental results from polarized electron-proton scattering [1].

**FIG. 2.** Calculated electric FF of the proton is compared with the data analysis of Ref. [17].

**FIG. 3.** Ratio of the generalized Sachs form factors (15) as a function of the momentum squared $p^2$ of the off-shell proton, for various fixed values of the photon momentum squared $Q^2 = -q^2$. The vertical dotted line corresponds to both nucleons being on-shell with $p^2 = m^2$. The cusp at $p^2 \approx 1.16 \text{ GeV}^2$ indicates the opening of the pion production threshold.
Although the half-off-shell FFs are not measurable by themselves [18], they can be an important part of various models for physical processes. For example, a two-photon exchange contribution to electron-proton scattering contains $\gamma NN$ vertices with an off-shell nucleon leg. So far the dependence of these vertices on the nucleon off-shell momentum has not been taken into account in the existing approaches [19]. Our calculation shown in Fig. 3 suggests that this dependence may be significant. It should be borne in mind, however, that in view of the representation dependence of Green’s functions [20], the calculation of such off-shell vertices should be consistent with the model for the physical processes to which they are applied.

V. CONCLUSIONS

In this report we extended the nonperturbative dressing procedure of the dressed $K$-matrix model (DKM) to calculate a $\gamma NN$ vertex with an off-shell nucleon and an off-shell photon. The principal motivation for this work is to study the nucleon electromagnetic interactions in a dynamical approach that describes in a unified manner two distinct types of reactions: those where essential contributions come from an off-shell nucleon (as in Compton scattering) and those where one probes the dependence on the momentum transferred by an off-shell photon (as in form factors extracted from electron-proton scattering experiments). Having previously applied DKM to Compton and $\pi N$ scattering, in this report we focused on the momentum-transfer dependence. We developed the dressing procedure to calculate the ratio of the proton electromagnetic form factors and found good agreement of our results with the recent JLab measurements.

The main limitation of the present version of DKM is the difficulty in describing the momentum-transfer dependence of the individual form factors as accurately as their ratio. To resolve this problem, additional dynamical contents—with attendant new parameters—might be required in the model. Rather than pursuing this direction in detail, we studied to what extent one can describe the dependence of the nucleon EM interactions on both the nucleon and photon variables using the same dynamical approach with a few predetermined parameters. Our approach is based on essential symmetry constraints, including relativistic and gauge invariance, unitarity, crossing, and causality, which are required in order to correlate in one model the very different kinematical regions explored in Compton and $\pi N$ scattering and in the proton form factors.

ACKNOWLEDGMENTS

We thank Shmuel Nussinov, Ralf Gothe, Pawel Mazur, and Peter Blunden for useful discussions. This work was supported in part by U.S. National Science Foundation Grant No. PHY-0140214. One of the authors (F.M.) appreciates the support and hospitality of the Nuclear Theory Group at Bonn University.