EXERCISES AND PROBLEMS
USC SUMMER SCHOOL 2012

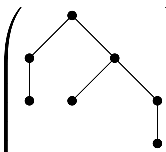
KAREN YEATS

1. Apology

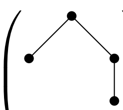
(1) Make a list of uncomplimentary things physicists say about mathematicians.
(2) Make a list of uncomplimentary things mathematicians say about physicists.

2. The Connes-Kreimer Hopf algebra of rooted trees

(1) What is

Δ ()

(2) What is

S ()

(3) What determines the signs of the terms in $S(T)$ for trees T?
(4) Can you give a nonrecursive characterization of S?
(5) Find all primitive elements of size 3.
(6) Find all primitive elements of size 4; can you find any general patterns?
(7) Here is a different combinatorial Hopf algebra. The underlying objects are strings of letters in some alphabet. The product is the shuffle, for example

$$abc \cdot xy = abcxy + abxcy + axbcy + xabcy + abxyc + axbyc + xabyc + axybc + xaybc + xyabc$$

The coproduct is deconcatenation, for example

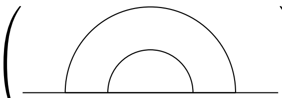
$$\Delta(abcde) = abcde \otimes 1 + abcd \otimes e + abc \otimes de + ab \otimes cde + a \otimes bcde + 1 \otimes abcde$$

Define $S(w) = -w - \sum_{uv=w} S(u) \cdot v$ where the sum is over ways to write w as the concatenation of u and v with u and v both having at least one letter.
   (a) Calculate S on a word of length 3.
   (b) Calculate S on a word of length 4.
   (c) What is true in general? Can you prove it?
   (d) Which words are primitive?

3. Renormalization Hopf algebras of Feynman graphs

(1) Renormalizability shows itself in this combinatorial context by the superficial degree of divergence depending only on the external structure. Show that property directly for the theories discussed, or your favourite renormalizable QFT.

(2) (a) Calculate

$\Delta\Bigg($  $\Bigg)$

in ϕ^3 in 6 dimensions.

(b) Calculate the coproduct of the graph with the same skeleton but with all gluon propagators in QCD.

(3) Show there is no two loop graph which is primitive in ϕ^4.

(4) Find a linear combination of two loop graphs in ϕ^4 which is primitive.

4. Combinatorial Dyson–Schwinger equations

(1) In rooted trees consider

$$X = \mathbb{I} - xB_+\left(\frac{1}{X}\right)$$

Expand and write $X = \mathbb{I} - \sum_{k\geq 1} x^k a_k$.

(a) Calculate $\Delta(a_k)$ for $k \leq 5$.

(b) Can you write $\Delta(a_k)$ for $k \leq 5$ using only the a_j $j \leq k$?

(c) Can you show this is possible for all k?

(2) (a) Write a combinatorial Dyson–Schwinger equation for the part of QED built by iterating the one loop vertex graph into itself in all ways.

(b) What is Q?

(c) Give a Dyson–Schwinger equation on rooted trees that has the same shape.

(d) What kind of trees does it generate?

(e) Repeat the first two parts also allowing the two loop vertex primitive

5. The abstract algebra behind B_+

(1) Show that $b^2 = 0$.

(2) Characterize maps $L : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ with the property that $L = bK$ for some $K : \mathcal{H} \to \mathcal{H}$.

(3) Consider

$$X = 1 - xB_+^{\frac{1}{2}\,-\!\bigcirc\!-}\left(\frac{1}{X^2}\right)$$

in ϕ^3. Calculate X to order $O(x^4)$.

(4) For each factor appearing in the definition of B_+ find some graphs for which that factor is 1 and some graphs for which that factor is not 1.

6. Green's functions and Slavnov-Taylor identities

(1) Suppose we have a scalar field theory with both a 3 and 4 valent coupling.
   (a) Work out both Qs.
   (b) Work out what the equivalent of the Slavnov-Taylor identity would say, expand it explicitly up to 3 loops.

(2) What does the Ward identity in QED say in this language.

7. Analytic Dyson–Schwinger equations

(1) Let R be the map taking Laurent series to Laurent series given by

$$R\Big(\sum_{j=-n}^{\infty} a_j x^j\Big) = \sum_{j=-n}^{-1} a_j x^j$$

Show

$$R(A)R(B) + R(AB) = R(R(A)B) + R(AR(B)).$$

This says that R is a Rota-Baxter operator and hence that minimal subtraction is a valid way to renormalize.

(2) Consider

$$X = 1 - xB_+^{\frac{1}{2}\,-\!\!\bigcirc\!\!-}\left(\frac{1}{X^2}\right)$$

in ϕ^3. Write down the corresponding analytic Dyson–Schwinger equation.

(3) Still in ϕ^3 modify the above to

$$X = 1 - xB_+^{\frac{1}{2}\,-\!\!\bigcirc\!\!-}\left(\frac{1}{X}\right)$$

where now we only allow insertion into the bottom propagator. Write down the corresponding analytic Dyson–Schwinger equation.

(4) In the single equation case suppose $\gamma_k(x) = \sum_{i\geq k}\gamma_{k,i}x^i$. Rewrite the recurrence for γ_k coming from the renormalization group equation using only the $\gamma_{\ell,j}$.

8. The Yukawa example

(1) Consider the differential equation

$$\gamma_1(x) = x - \gamma_1(x)\left(1 - sx\frac{\partial}{\partial x}\right)\gamma_1(x)$$

   (a) What does changing s do to the qualitative behaviour of the differential equation?
   (b) For which values of s can you get a solution (explicit or implicit)?
   (c) Consider more generally

$$\gamma_1(x) = P(x) - \gamma_1(x)\left(1 - sx\frac{\partial}{\partial x}\right)\gamma_1(x)$$

   Can you find any functions P other than $P(x) = x$ for which there are exact solutions for some values of s?

(2) Broadhurst and Kreimer using cleverness also considered the analytic version of

$$X = 1 - xB_{+}^{\frac{1}{2}-\!\!\bigcirc\!\!-}\left(\frac{1}{X}\right)$$

in ϕ^3 where we only allow insertion into the bottom propagator. They found a higher order differential equation for the anomalous dimension. With the benefit of hindsight what can you do?

9. Rewriting the Dyson–Schwinger equations

(1) In the example of reducing to one insertion place calculate q_4, the error at order 4, and check that q_4 is primitive.
(2) Write a computer program to calculate the coefficients of P as functions of the coefficients of the $F_k(\rho)$ given a value of s. Start with just $k = 1$.

10. Pictures of the differential equations

(1) In the $P > 0$ case, for example in QED, consider the solutions which go to -1 as $L \to \infty$. What would these solutions mean physically? Do they seem physically realistic to you or not?
(2) In the $P < 0$ case, for example in QCD, consider the solutions which go to ∞ as $L \to \infty$ and go to $-\infty$ as $L \to -\infty$ only existing for x sufficiently large (these ones are the C-shaped ones). What would these solutions mean physically? Do they seem physically realistic to you or not?
(3) In the $P < 0$ case, for example in QCD, there is something special about solutions which are linear for large x.
  (a) What would such solutions mean physically?
  (b) Such solutions diverge in finite L, however in projective space one can view a line going out at a particular slope as reappearing in the opposite quadrant coming in with the same slope. What physical sense can you make of this idea?
(4) Consider the following system

$$\frac{d\gamma_1^+}{dL} = \gamma_1^+ - (\gamma_1^+)^2 - P^+$$

$$\frac{d\gamma_1^-}{dL} = \gamma_1^- + (\gamma_1^-)^2 - P^-$$

$$\frac{dx}{dL} = x(\gamma_1^+ + 2\gamma_1^-)$$

What qualitatively different behaviours are possible for solutions to this system? The most interesting case is $P^- < 0$ and $P^+ > 0$ as this corresponds to massless ϕ^4.

11. A chord diagram expansion

(1) For all connected rooted chord diagrams with at most four chords work out the corresponding monomials in γ_1, γ_2, γ_3, and γ_4.
(2) What happens for $s = 3$ or $s = -1$?