USC Summer school Aug 3-6 2012 | Karen Yeats

① Apology

② Connes-Kreimer Hopf alg of rooted trees w/ B^+ (Hopf sem begining of lec1 w/ end of lec 2)

③ Renormalization Hopf algs of Feynman graphs (Hopf sem lec 3 but not the Hochschild coh

④ Tree recurrences w/ comb. DSEs (Hopf sem lec 4 + recent talks)

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⑤ Hochschild coh + B_+ as a 1 cocycle ↙ put universality here (Bergbauer-Kreimer + Anatomy)

⑥ coproduct on Greens functions + Slavnov-Taylor (van Suijlekom chapter 4

⑦ Also briefly what does renormalization look like in this language.
Getting to my analytic setup (not a whole hour!!!)
+ RGE giving first recurrence
(emphasize first recurrence)

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⑧ The Yukawa eg all the way through (some old notes + Broadhurst-Kreimer)

⑨ also phrasing it my inside out way
Reduce to 1 insertion place w/ geometric series w/ leave second recurrence.

⑩ Pictures of the DE eqn in QCD

↑ Or flatten it up with
the 2 different places
to find transcendentals.
or put places to find
transcendentals as an extra
hour on the last day

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⑪ Chord diagram expansion.

⑫

⑬

USC summer school Aug 3-6, 2012

① Apology

- To start I'll be perturbative and a mathematician ultimately say some nonperturbative thing of physical interest
- overview Hopf algs & comb DSEs
  Hochschild cohomology, Green functions & Slavnov-Taylor
  Analytic DSEs and a reduction
  a DE for QCD & a chord diagram expansion

② The Connes-Kreimer Hopf alg of rooted trees

2.1 rooted trees

egs, not plane

$\mathbb{1}$ = empty tree

A pulling apart operator $\Delta(T) = \sum_{\text{adm } c} P_c(T) \otimes R_c(T)$

define admissible cut, $P_c(T)$, $R_c(T)$ with egs
note empty cut and full cut

eg $\Delta($  $)$

2.2 Hopf algebras

whenever you have some comb. objs & a pulling apart operator you should think Hopf alg

Make trees an alg by
- prod is disj union
- + & scalar mult is formal
- unit is $\mathbb{1}$

Note unit tells how to find the underlying field inside the alg.

Coalgs are the dual of algs
instead of a product wich glues elements together
there is a coproduct which pulls them apart
instead of a unit which tells how to find the field in the alg
there is a counit which puts the alg in the field

Make trees a coalg by

- Δ
- $\varepsilon(\mathbb{1}) = 1$, $\varepsilon(T) = 0$

In fact this is a Hopf alg which means

- the co structure plays nicely with the alg str (Δ, ε are alg homomorphisms)
- There is an antipode

$$S(T) = -T - \sum_c S(P_c(T))\, R_c(T)$$

eg

$$S(\bullet) = -\bullet$$

$$S(\vert) = -\vert - (S(\bullet)\bullet) \qquad \Delta(\vert) = \mathbb{1}\otimes\vert + \vert\otimes\mathbb{1} + \bullet\otimes\bullet$$

$$= -\vert + \bullet\bullet$$

$$S(\Lambda) = -\Lambda - 2S(\bullet)\vert - S(\bullet\bullet)\bullet \qquad \Delta(\Lambda) = \mathbb{1}\otimes\Lambda + \Lambda\otimes\mathbb{1} + 2\bullet\otimes\vert + \bullet\bullet\otimes\bullet$$

$$= -\Lambda + 2\bullet\vert - S(\bullet)^2\bullet$$

$$= -\Lambda + 2\bullet\vert - \bullet\bullet\bullet$$

An element a is **primitive** in a Hopf alg if $\Delta(a) = \mathbb{1}\otimes a + a\otimes\mathbb{1}$

which trees are primitive? answer $\Delta(\bullet) = \bullet\otimes\mathbb{1} + \mathbb{1}\otimes\bullet$
nothing else

are any other elements of the Hopf alg primitive?
answer yes $2\vert - \bullet\bullet$

The **size** of a tree or forest is the number of vertices

eg $\vert\,\bullet$
forest of size 3

(p 3)

2.3 B_+

The power of trees is that they have a natural recursive structure – a tree is built out of subtrees

T :


Call the add-a-root operator B_+

$B_+(T_1 \cdots T_n) =$


eg $B_+(| \cdot) = \Lambda$

How does B_+ relate to the Hopf alg?

eg consider $T = \wedge$ what is $\Delta(B_+(T))$?

$$\Delta(B_+(\wedge)) = \Delta(\lambda) = \lambda \otimes \mathbb{1} + \mathbb{1} \otimes \lambda + 2 \cdot \otimes | + \cdot\cdot \otimes | + \wedge \otimes \cdot$$

compare $\Delta(\wedge) = \wedge \otimes \mathbb{1} + \mathbb{1} \otimes \wedge + 2 \cdot \otimes | + \cdot\cdot \otimes \cdot$

what is the correction?

$$\Delta(B_+(\wedge)) = \lambda \otimes \mathbb{1} + (\mathrm{id} \otimes B_+)\, \Delta(\wedge)$$

More generally if T is any tree

$$\Delta(B_+(T))$$

$$= \underbrace{B_+(T) \otimes \mathbb{1}}_{\text{full cut}} + \sum_{\substack{\text{"true" adm} \\ \text{cuts} \\ C \text{ of } B_+(T)}} P_c(B_+(T)) \otimes R_c(B_+(T))$$

$$= B_+(T) \otimes \mathbb{1} + \sum_{\substack{\text{"true" adm} \\ \text{cuts of } B_+(T) \\ = \text{any adm} \\ \text{cut of } T}} P_c(T) \otimes R_c(B_+(T))$$

$$= B_+(T) \otimes \mathbb{1} + \sum_{\substack{\text{adm } C \\ \text{of } T}} P_c(T) \otimes B_+(R_c(T))$$

$$= B_+(T) \otimes \mathbb{1} + (\mathrm{id} \otimes B_+)\, \Delta(T)$$

Similarly for a forest

$$\Delta(B_+(F)) = B_+(F)\otimes 1 + \sum_{\substack{\text{"the adm}\\ \text{cuts } C\\ \text{of } B_+(F)}} P_C(B_+(F))\otimes R_C(B_+(F))$$

$$= B_+(F)\otimes 1 + (id\otimes B_+)\prod_{\substack{T \text{ tree}\\ \text{of } F}}\ \sum_{\substack{\text{adm}\\ \text{cuts}\\ \text{of } T}} P_C(T)\otimes R_C(T)$$

$$= B_+(F)\otimes 1 + (id\otimes B_+)\Delta(F)$$

So $\boxed{\Delta(B_+(F)) = B_+(F)\otimes 1 + (id\otimes B_+)\Delta(F)}$

We'll return to this

2.4 What, why, refs

This is the Connes-Kreimer ren. Hopf alg.

we trees to represent structure of subdivs in a Feynman graphs

Then S tells how to renormalize (via forest formula)

refs Connes - Kreimer arXiv: hep-th/9912092
CMP 210 (2000) 249-273
Kreimer arXiv: q-alg/9707029
Adv. Theo. Math Phys 2: 303-334 (1998)

③ Renormalization Hopf algs of Feynman graphs

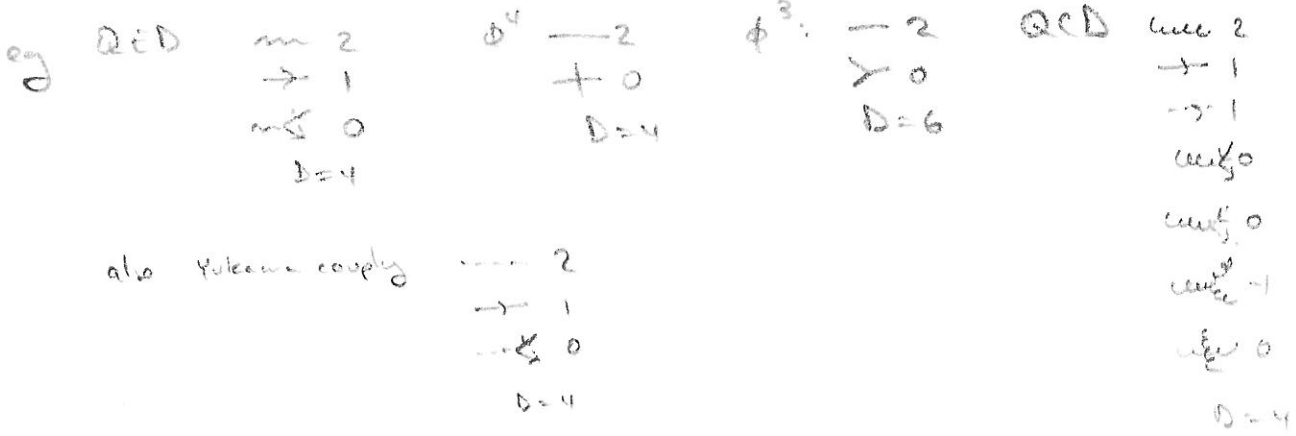
3.1 What to tell a graph theorist

To tell a graph theorist about Feynman diagrams build graphs of half-edges and vertices

- pairs of half edges form internal edges
- unmatched half edges are external edges

To build the ren. Hopf alg we only need the following input from the physical theory

- A set of permissible edge types
- A set of permissible vertex types
- power counting weights for each type
- D dim of space time.



Aside on labelled vs unlabelled counting

in pQFT external edges are labelled while internal are unlabelled

However in perturbative expansions graphs are weighted by symmetry factors $\frac{1}{|\mathrm{Aut}\, G|}$

So we are actually doing labelled counting (on the half edge) with exponential generating functions and then forgetting the labels for internal edges

This is also why go from full greens fn to connected greens fn by exp

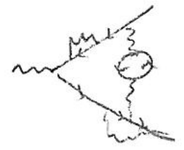
The edge and vertex weights are all we need to recognize UV divergent graphs by power counting

Define the superficial degree of divergence of a graph in a theory to be

$$D\ell - \sum_{\substack{e \\ \text{int}}} w(e) - \sum_{v} w(v)$$

if ≥ 0 say graph is div.

eg what are all the div. subgraphs of



in QED. Allow any 1PI subgraphs and disjoint unions of 1PI subgraphs

eg calculate div of

0

,

0

,

0

yay still renormalizable.

3.2 Hopf algebras of Feynman graphs

We want to build a Hopf alg again

As before the alg structure is somewhat trivial

Take 1PI divergent Feynman graphs in a theory

For technical reasons (want the Hopf alg to be connected) normalize all the tree level (but with w isolated propagators since 1PI) to 1

Mult is disj union

\+ is formal

unit 1 is "empty graph"

The interesting map is the coproduct

(p7)

$$\Delta(G) = \mathbb{1} \otimes G + G \otimes \mathbb{1} + \sum_{\substack{\gamma \subseteq G \\ \gamma \text{ prod of 1PI div sub}}} \gamma \otimes G/\gamma$$

← contract each component to the vertex or edge given by its external str.

eg $\Delta($  $)$ in QED

eg $\Delta($  $)$ in QED

eg $\Delta($  $)$ in ϕ^4

Analogously get the antipode

$$S(G) = -G - \sum_{\gamma} S(\gamma)\, G/\gamma$$

Define primitives as before.

lets find some primitives in QED
any 1 loop graph must be prim:

 ,  , 

are there more? yes of course

 , 

& lots more

are there any more propagator corrections which are prim?

no

 subs div provid not $\mathbb{1}$.

3.3 Graphs to trees and back again

In simple cases subdivergences have a tree structure

eg 

eg 

This tells us what B_+ needs to be → insertion

However there are subtleties (see tomorrow)

There are two useful notions of size

- loop order
- Augmentation degree = # of vertices in insertion tree

eg 

loop order 3

aug deg 2

3.4 Why, refs

This is arguably more natural for QFT than the trees (though we'll see the trees are universal in a precise sense)

ref : my thesis for the comb setup

Now a memoirs AMS

or arXiv : 0810.2249

④ Combinatorial DSEs

4.1 tree recurrences

recall B_+

Consider $X = \mathbb{1} + x B_+(X)$

- expand
- what is the general pattern
- write $X = \sum x^n c_n$ what does $\Delta(c_n)$ look like

Consider $X = \mathbb{1} + x B_+(X^2)$

same questions get

$$\Delta(c_1) = c_1 \otimes c_0 + c_0 \otimes c_1$$
$$\Delta(c_2) = c_2 \otimes c_0 + c_0 \otimes c_2 + 2c_1 \otimes c_1$$
$$\Delta(c_3) = c_3 \otimes c_0 + c_0 \otimes c_3 + 3c_1 \otimes c_2 + (c_1^2 + 2c_2) \otimes c_1$$
$$\Delta(c_4) = c_4 \otimes c_0 + c_0 \otimes c_4 + (2c_3 + 2c_1c_2) \otimes c_1 + (3c_1^2 + 3c_2) \otimes c_2 + 4c_1 \otimes c_3$$

Consider $X = \mathbb{1} - x B_+\left(\frac{1}{X}\right)$

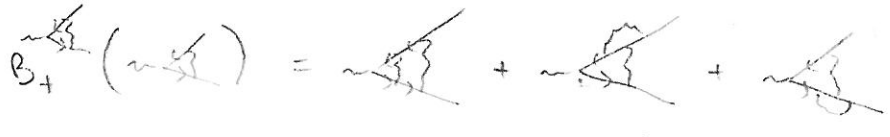
expand and what does it count but leave closed under Δ to exercises.

The fact that $\Delta \circ B_+ = B_+ \otimes \mathbb{1} + (id \otimes B_+) \circ \Delta$ is key as we'll see tomorrow

4.2 Combinatorial DSEs for Feynman graphs

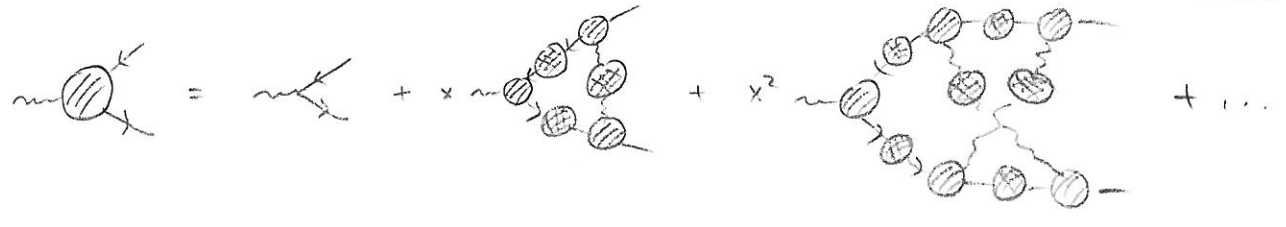
Recall B_+^{γ} is insertion into the prim graph γ

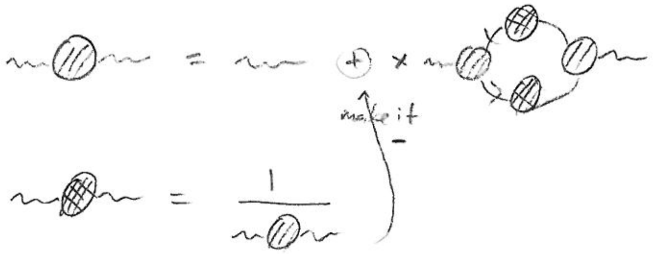
eg

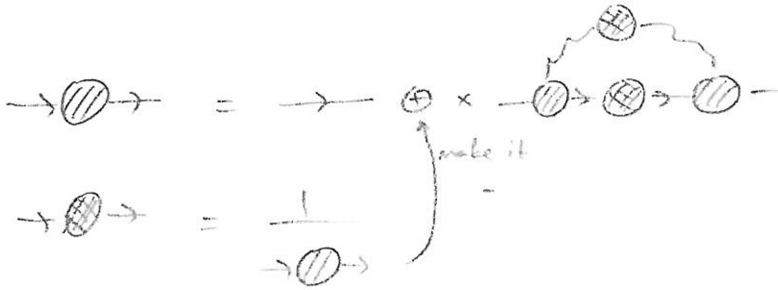


all possible ways (for details of weights see tomorrow)

Write out QED expansion in terms of blobs and skeletons







skeletons are primitives inserting into B_+ so rewrite

$$X^{\sim<} = 1 + x B_+\left(\frac{(X^{\sim<})^3}{(X^{\to})^2 X^{\sim}}\right) + x^2 B_+\left(\frac{(X^{\sim<})^5}{(X^{\to})^4 (X^{\sim})^2}\right) + \dots$$

$$X^{\sim} = 1 - x B_+\left(\frac{(X^{\sim<})^2}{(X^{\to})^2}\right)$$

$$X^{\to} = 1 - x B_+\left(\frac{(X^{\sim<})^2}{X^{\to} X^{\sim}}\right)$$

	Answers
now how many vertex insertion places does a 3 loop vertex skeleton have?	7
k loop?	2k+1
how many photon insertion places?	k
how many fermion insertion places?	2k

$$X^{\sim<} = \mathbb{1} + \sum_k x^k B_+^{\text{loopk prims}}\left(X^{\sim<} \underbrace{\left(\frac{(X^{\sim<})^2}{X^{\sim}(X^{\to})^2} \right)^k}_{} \right)$$

can we
see this in $X^{\sim}$ & $X^{\to}$?
call it Q

yes $$X^{\sim} = \mathbb{1} - x B_+^{\sim\bigcirc\sim}\left(X^{\sim} Q \right)$$

$$X^{\to} = \mathbb{1} - x B_+^{\overset{\frown}{\to}}\left(X^{\to} Q \right)$$

$$X^{\sim<} = \mathbb{1} + \sum_k x^k B_+^{\substack{\text{loop k} \\ \sim< \text{ prims}}}\left(X^{\sim<} Q^k \right)$$

call Q the combinatorial invariant charge. This pattern is very general in physical examples

So for me the general form of a system of comb DSEs is

$$\boxed{\begin{array}{l} X^r = \mathbb{1} + \operatorname{sgn}(s_r) \sum_{k \geq 1} x^k B_+^{k,r}\left(X^r Q^k \right) \\ \text{where } Q = \prod_r (X^r)^{-s_r} \end{array}}$$

match all the parts with what is above. (what are the r
what are the s_r ← what does s_r mean

Systems get messy so often I'll take single equations to clarify matters
Similar results hold for systems but they are messier.

$$\boxed{X = \mathbb{1} \pm \sum_{k \geq 1} x^k B_+^k \left(X Q^k \right) \qquad \text{where } Q = X^{-s}}$$

eg Broadhurst-Kreimer bit of Yukawa theory

- comb DSE
- graphs

4.3 why, refs

These equations (comb DSEs) are our first step from graphs to nonperturbative physics. The comb DSEs are simply built from the recursive nature of the graphs themselves. Yet their analytic avatars are DSEs in the usual sense and here can see the nonperturbative world. Today though we remained combinatorialists and just played with trees and graphs

refs Kreimer - I (06 LL proceedings) hep-th/0605096

Broadhurst-Kreimer hep-th/0012146

nucl. Phys B

600 (2001) 403-422

(graphs also 0810.2279)

(5) The abstract algebra behind B_+

5.1 Hochschild cohomology

Today we'll get away from pure combinatorics to begin matters will get (worse) with some abstract algebra

but then we'll finally become more <u>analytic</u> and more <u>physical</u>

let $\mathcal{H}$ be a bialgebra (ie an alg and a coalg and they play well together)

let $L : \mathcal{H} \longrightarrow \overbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}^{n \text{ times}}$ be a linear map

define $bL : \mathcal{H} \to \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n+1 \text{ times}}$

$$\text{by} \quad bL = (id \otimes L)\Delta + \sum_{i=1}^{n} (-1)^i \Delta_i L + (-1)^{n+1} L \otimes \mathbb{1}$$

$$\Delta_i = id \otimes \cdots \otimes id \otimes \underset{\substack{\uparrow \\ i^{th} \\ \text{slot}}}{\Delta} \otimes id \otimes \cdots \otimes id$$

eg say $L : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$

$$\begin{aligned} bL &= (id \otimes L)\Delta + (-1)^1 (\Delta \otimes id) L + (-1)^2 (id \otimes \Delta) L + (-1)^3 L \otimes \mathbb{1} \\ &= (id \otimes L)\Delta - (\Delta \otimes id) L + (id \otimes \Delta) L - L \otimes \mathbb{1} \end{aligned}$$

sub eg now say L takes a binary tree (with ordered children) to the pair of its left and right children

eg $L(\Lambda) = \bullet \otimes \Lambda$.

$$\begin{aligned} \text{then} \quad bL(\Lambda) &= (id \otimes L)\Delta(\Lambda) - (\Delta \otimes id) L(\Lambda) + (id \otimes \Delta) L(\Lambda) - L(\Lambda) \otimes \mathbb{1} \\ &= (id \otimes L)(\Lambda \otimes \mathbb{1} + \mathbb{1} \otimes \Lambda + \bullet \otimes \backslash + \bullet \otimes /) \\ &\quad - (\Delta \otimes id)(\bullet \otimes \bullet) + (id \otimes \Delta)(\bullet \otimes \bullet) - \bullet \otimes \bullet \otimes \mathbb{1} \\ &= \Lambda \otimes \mathbb{1} \otimes \mathbb{1} + \cancel{\mathbb{1} \otimes \bullet \otimes \bullet} + \bullet \otimes \mathbb{1} \otimes \bullet + \bullet \otimes \bullet \otimes \mathbb{1} \\ &\quad - \cancel{\bullet \otimes \mathbb{1} \otimes \bullet} - \cancel{\mathbb{1} \otimes \bullet \otimes \bullet} + \cancel{\bullet \otimes \mathbb{1} \otimes \bullet} + \cancel{\bullet \otimes \bullet \otimes \mathbb{1}} - \cancel{\bullet \otimes \bullet \otimes \mathbb{1}} \\ &= \Lambda \otimes \mathbb{1} \otimes \mathbb{1} + \bullet \otimes \mathbb{1} \otimes \bullet + \bullet \otimes \bullet \otimes \mathbb{1} \end{aligned}$$

As it turns out $b^2 = 0$ (ie $b(b(L)) = 0$ fn for all L)
(see exercises)

What this tells us is that we can build a coh. theory.
called Hochschild coh.

3 line summary of cohomology

① You need maps

$$\begin{tikzcd} {} \arrow[r, "b"] & C_n \arrow[r, "b"] \arrow[rr, bend left, "b^2"] & C_{n+1} \arrow[r, "b"] & C_{n+2} \arrow[r, "b"] & \cdots \end{tikzcd}$$

with $b^2 = 0$

≀

"things of size n"

(in our case the things of size n are lin map $L : \mathcal{H} \to \mathcal{H}^{\otimes n}$)

② You form quotients (gps, vs., etc as applicable)

$$\underbrace{\ker b / \operatorname{im} b}$$ for each n

③ You use these to understand whatever you started with

5.2 B_+ as a 1 cocycle in rooted trees

The 3 line summary of coh says that the first important thing is the kernel of the first b

that is if $L : \mathcal{H} \to \mathcal{H}$

what does $bL = 0$ mean?

$$0 = bL = (id \otimes L)\Delta - \Delta L + L \otimes \mathbb{1}$$

so $\Delta L = L \otimes \mathbb{1} + (id \otimes L)\Delta$

This was the B_+ identity (check)

This says B_+ on rooted trees is a 1-cocycle

This is all the cohomology we'll need.

In fact B_+ on rooted trees is in a precise sense universal

Consider any pair (A, L) of a commutative bialgebra A and a Hochschild 1 cocycle L on A

There is always a map

$$\rho : (\mathcal{H}, B_+) \longrightarrow (A, L)$$

p15

given by $\rho(\mathbb{1}) = \mathbb{1}$

and then $\rho(B_+(a)) = L(\rho(a))$

and $\rho(ab) = \rho(a)\rho(b)$

Thus any time we have a 1 cocycle in a fundamental sense B_+ sits inside it

5.3 B_+ as a 1-cocycle in Feynman graphs

In view of this (and more) we would like B_+^γ

γ Feynman graph to also have the 1 cocycle property:

$$\Delta B_+ = (id \otimes B_+)\Delta + B_+ \otimes \mathbb{1}$$

First this tells us γ must be primitive

otherwise if $\Delta\gamma = \gamma\otimes\mathbb{1} + \mathbb{1}\otimes\gamma + \gamma'\otimes\gamma''$

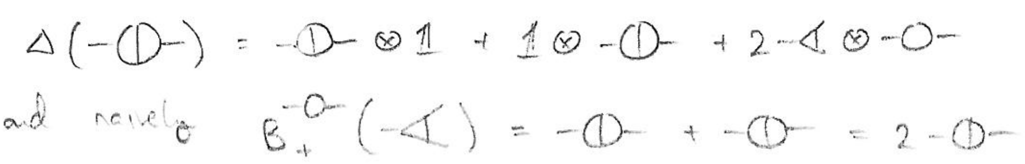
then $\Delta B_+^\gamma(\mathbb{1}) = \Delta\gamma = \gamma\otimes\mathbb{1} + \mathbb{1}\otimes\gamma + \gamma'\otimes\gamma'' \;\not\!=$

but $(id\otimes B_+^\gamma)\Delta(\mathbb{1}) + (B_+^\gamma\otimes\mathbb{1})\,\mathbb{1} = id\otimes\gamma + \gamma\otimes\mathbb{1}$

We already said if there are different ways to insert then just sum over them

But sometimes get double counting from overlapping divergences

eg  in ϕ^3 in 6 dim



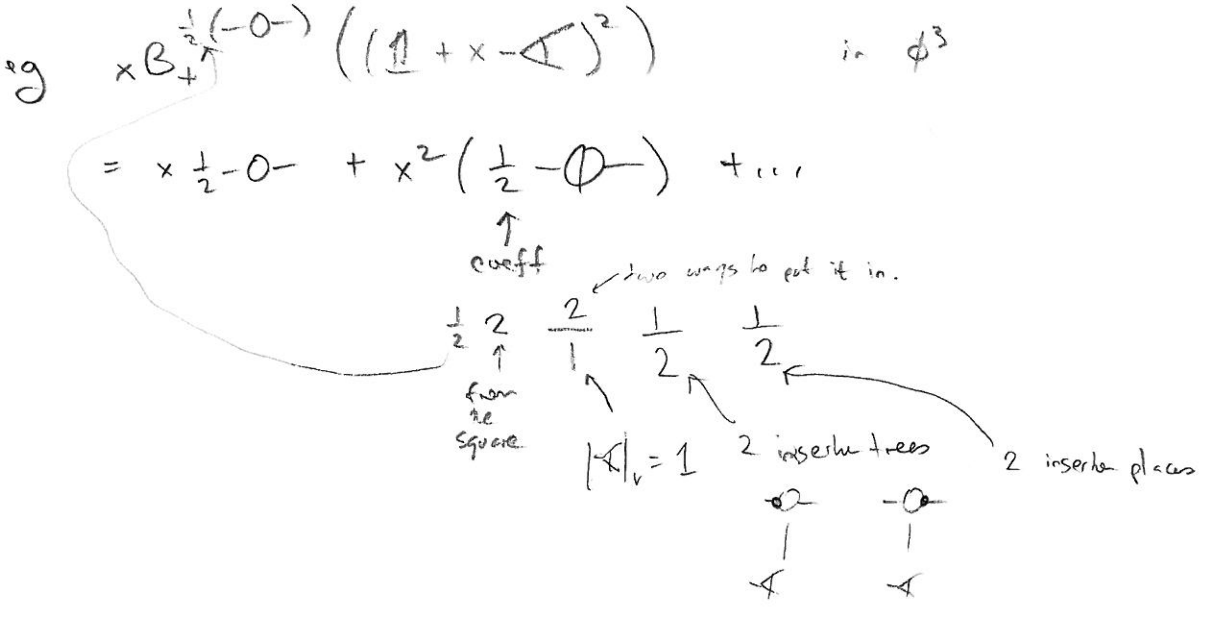
but we dont want to overcount

so insert a nasty coefficient to solve this problem

$$B_+^{\gamma}(G) = \sum_{\substack{\Gamma \text{ Feynman} \\ \text{graph in} \\ \text{theory in} \\ \text{question}}} \frac{\text{bij}(\gamma, G, \Gamma)}{|G|_v} \frac{1}{\text{maxf}(\Gamma)} \frac{1}{(\gamma|G)} \Gamma$$

where
- maxf(Γ) is the # of insertion trees corresponding to Γ
- $|G|_v$ is the # of distinct graphs obtainable by permuting the external edges of G
- bij(γ, G, Γ) is the # of bij of the external edges of X with an insertion place of γ s.t. the resulting insertion gives Γ
- (γ|G) is the # of insertion places for X in γ

This was constructed specifically so that if we sum over all primitives with a given external structure and insert into all insertion places of each then we get all 1PI graphs with that external structure weighted by their symmetry factor



But there are more problematic kinds of overlaps

consider  in QCD

This can be obtained by inserting  into 

or by inserting  into 

so  $\otimes$  appears in ΔB_+  ()

but not in $(id \otimes B_+$  $)\Delta + B_+$  $\otimes \mathbb{1}$ ()

$=$  $\otimes$  $+ 1 \otimes B_+$  ()

$+ B_+$  () $\otimes \mathbb{1}$

no change of the coefficient can save this.

How do we solve it? Physics! The Slavnov-Taylor identities give identities between graphs which fix this problem as we'll see in more detail in the next lecture

5.4 Why and refs

All this algebra is the stepping stone to getting the physics in

refs. Bergbauer-Kreimer IRMA lect Math Theor Phys 10 (2006) 133-164
arXiv: hep-th/0506190
Kreimer Annals Phys 321 (2008) 2757-2781
arXiv: hep-th/0509135

⑥ Green's functions and Slavnov-Taylor identities

6.1 Green's functions and the coproduct.

But graphs aren't really the right object to be working with - especially at a non perturbative summer school

They aren't physical

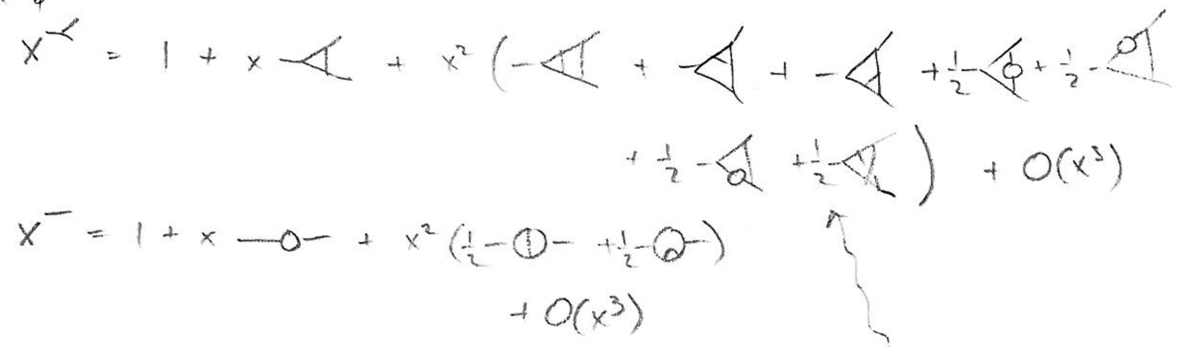
A step closer (while staying combinatorial) are the (comb) 1PI Green's functions

$$X^r = 1 \pm \sum_{\substack{\Gamma \text{ with} \\ \text{external} \\ \text{struct } r}} \frac{\Gamma}{\mathrm{sym}(\Gamma)} x^{|\Gamma|}$$

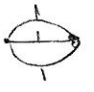
loop order

acting as a coupling (or can just leave it out)

eg in ϕ^3 ← to get a few non triv sym. factors (as opposed to QED) but stay simple



why is this $\frac{1}{2}$? another way to draw the graph



swap left to right

Say we do this for a theory with 1 vertex

let $$Q = \left(\frac{(X^v)^2}{\underset{\substack{\text{edges e} \\ \text{making} \\ \text{up v} \\ \text{(with} \\ \text{multiplicity)}}}{\prod} X^e}\right)^{\boxed{\frac{-1}{\mathrm{val}(v)-2}}}$$

why? this is the correct normalization as we'll see later

eg $$Q_{QCD} = \left(\frac{(X^{\sim\!\prec})^2}{X^{\sim}(X^{-})^2}\right)^1 \qquad Q_{\phi^3} = \frac{(X^{\prec})^2}{(X^{-})^3} \qquad Q_{\phi^4} = \left(\frac{(X^{+})^2}{(X^{-})^4}\right)^{\frac{1}{2}}$$

we already saw this in the DSE

$$= \frac{X^{+}}{(X^{-})^2}$$

Then $$\Delta([x^k]X^r) = \sum_{j=0}^{k} [x^j]X^r Q^{k-j} \otimes [x^{k-j}]X^r$$

$$\Delta([x^k]X^rQ^l) = \sum_{j=0}^{k} [x^j]X^rQ^{k+l-j} \otimes [x^{k-j}]X^rQ^l$$

where $[x^k]P$ means the coefficient of x^k in the power series P

equivalently it is projection onto the loop k part of P

This says that we can make sense of the coproduct on the whole Green's function, namely

$$\Delta(X^r) = \sum_l X^rQ^l \otimes \underbrace{x^l([x^l]X^r)}_{\text{another way to say to project onto loop } l \text{ part but keep power of coupling}}$$

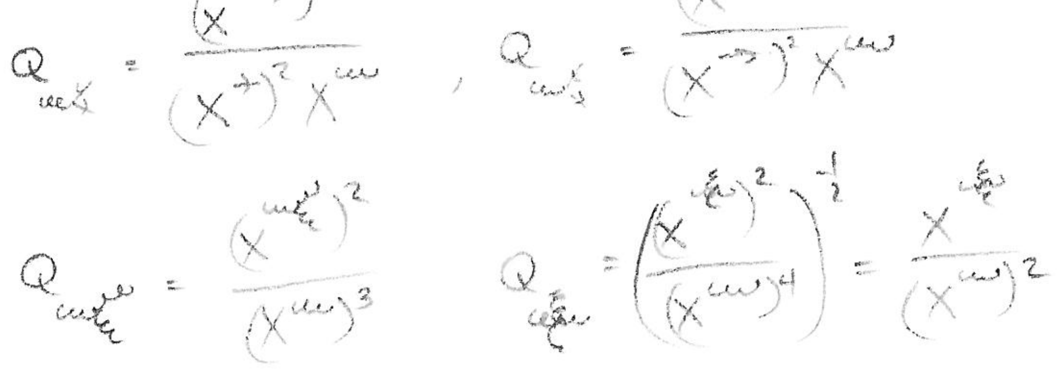
put the monomial back in

6.2 Multiple vertices and the Slavnov-Taylor identities

But what if we have more than one vertex in our theory?

Then we get more than one Q

eg in QCD



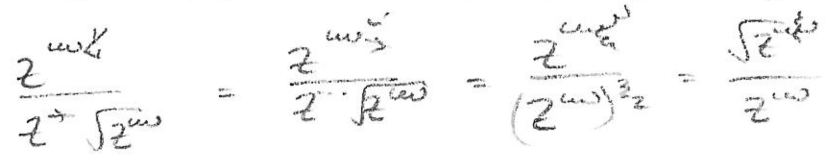
The Green's function still has a sensible coproduct

(van Suijlekom) $$\Delta X^r = \sum_{n_1 \dots n_k} X^r \prod_v Q_{v_i}^{n_i} \otimes \underbrace{\mathrm{Proj}_{\frac{2n_1}{\mathrm{val}\, v_1 - 2} \cdots \frac{2n_2}{\mathrm{val}\, v_2 - 2} \cdots}}_{\text{project onto exactly this many of each vert}} (X^r)$$

But something even ~~better~~ is going on

At the level of the Z-factors for renormalization of QCD

we know

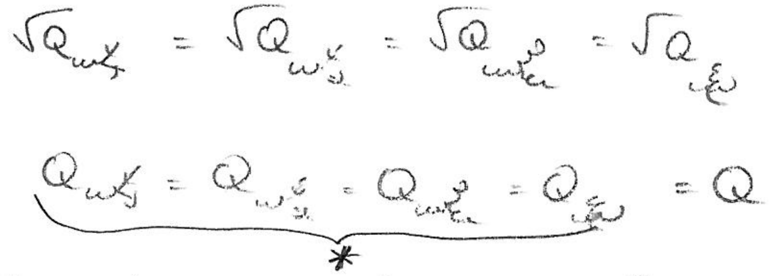


Slavnov-Taylor id's

We can think of the Z-factors as the result of building the counter terms out of each graph in the corresponding X^r

so $Z^r = C(X^r)$

So at the graph level the Slavnov-Taylor ids are



(and that's why the val v_i - 2 stuff)

So really there is only one Q

The precise mathematical statements are (van Suijlekom)

(a) The ideal J generated by * in the algebra of Feynman graphs is in fact a Hopf ideal

that is $\Delta J \subseteq J \otimes \mathcal{H} + \mathcal{H} \otimes J$

$\uparrow \quad \uparrow$ the Hopf alg

(p21)

(b) Hence since renormalization can be implemented using the Hopf alg, it is also valid in $\mathcal{H}/J$ and so the Slavnov-Taylor ids are compatible with renormalization

6.3 Back to B_+

Recall the remaining problem we had with B_+ not being a 1-cocycle was that

 could be built from

 in 

or from  in 

But Q  $=$ Q  so we have identities between just such graphs

and these are exactly what we need to make B_+ a cocycle so

$$\Delta B_+^\gamma = B_+^\gamma \otimes \mathbb{1} + (\mathrm{id} \otimes B_+^\gamma)\Delta \qquad \text{in } \mathcal{H}/J$$

for γ prim

In fact these two things match up very nicely if we started by demanding B_+^γ be a cocycle then we could most naturally do this by exactly imposing Slavnov-Taylor (can make unnatural examples where there are other options so this is not strictly iff but "physically" iff)

This also means that our comb. DSEs really are in the form

$$X^r = \mathbb{I} + \mathrm{sgn}(s_r) \sum_{k \geq 1} x^k B_+^{k,r}(X^r Q^k)$$

$$Q = \prod_r (X^r)^{-s_r}$$

in physical theories → even with multiple vertices

6.4 Why and refs

There are a few different ways to think about this depndg on what you view as comg first.

If we take the Slavnov-Taylor ids as known from QFT then we derive that the B^{k} are cocycles & hence that the comb DSEs have nice propertis

If we assume the comb DSEs are nice we then want the B^{k} to be cocycles and so naturally obtain Slavnov-Taylor

Once we know the coprod on Green's fncts the DSEs fall out

ref. first obs. of this in Kreimer Annals Phys 321 (2008) 2757-2781

arXiv: hep-th/0509135

Mathematical formulation in van Suijlekom in QFT (2008) arXiv: 0801.3170

see also a nice exposition in his lectures from the 2010 Les Houches workshop on Structures in local Quantum Field Theories

⑦ Analytic Dyson-Schwinger equations

7.1 Renormalization

So what does renormalization look like in this language

Usually Feynman diagrams index the perturbative expansion
each one stands for its Feynman integral

Another way to say this is we have a map
ϕ from graphs to formal integrals
ϕ is the Feynman rules

Then what do you do?

- You regularize typically
  say analytic reg
  or Dim reg
- You choose how to fix a single divergence
  eg minimal subtraction
  removes the pole part of a Laurent series
  call this map R
- Then take care of subdivergences
  the Hopf alg does this

let S_R^ϕ be

$$S_R^\phi(\mathbb{1}) = 1$$

$$S_R^\phi(G) = -R(\phi(G)) - \sum_{\gamma \subseteq G} S_R^\phi(\gamma) R(G/\gamma$$

(essentially S, but
now on the analytic side)

tells us the counter terms
and the renormalized Feynman rules
are $\phi_R = S_R^\phi * \phi$

makes sense intuitively
apply Feynman rules to the
outer part of the graph
and for the rest subtract
off recursively.

where $A * B$ is the map

$$\underset{\underset{\text{mult}}{\uparrow}}{m}(A \otimes B)\Delta$$

so pull apart apply A and B and put
back together.

(p24)

7.2 Analytic Dyson-Schwinger equations

Recall our combinatorial DSEs were

$$X^r = \mathbb{1} - \text{sgn}(s_r) \sum_k x^k B_+^{k,r}(X^r Q^k)$$

$$Q = \prod_r (X^r)^{-s_r}$$

By design of B_+ X^r is the combinatorial $1PI$ Green's function for external structure r

So if we apply Feynman rules then $\phi(X^r)$ is the honest-to-goodness analytic DSE → a recursive eqn for the honest Green's functions

Lets see an eg (Broadhurst-Kreimer)

$$X = \mathbb{1} - x B_+^{\frown}\left(\frac{1}{X}\right)$$

so as we saw this generates graphs like ...

will renormalize by subtracting at a fixed value of the external momenta
the DSE will handle the recursive subtractions we just need the outermost one

$$\boxed{G(\alpha, L) = 1 - \frac{\alpha}{q^2}\int d^4k \frac{k \cdot q}{k^2 G(\alpha, \log\frac{k^2}{\mu^2})(k+q)^2} - \cdots \Big|_{q^2=\mu^2}}$$

where $L = \log\frac{q^2}{\mu^2}$

This is a slightly unusual normalization of the usual form of the DSE for this circumstance:

$$q^2\Sigma(q^2) = \frac{a}{\pi^2}\int \frac{d^4k}{k^2 - k^2\Sigma(k^2)} \frac{q \cdot k}{(k+q)^2} - \text{subtractions so that } \Sigma(\mu^2) = 0$$

$$k^2(1 - \Sigma(k^2))$$

so $G = 1 - \Sigma$

$\alpha = \frac{a}{\pi^2}$

General shape will be the similar

$$G^r(x, \underset{\substack{\uparrow \\ \text{kinematical} \\ \text{parameters}}}{\cdots}) = 1 - \sum x^k \int \underset{\substack{\uparrow \\ \text{factors} \\ \text{coming from} \\ \text{vertices and propagators} \\ \text{of the skeleton}}}{\vdots} \prod_t G^t(x, \underset{\substack{\uparrow \\ \text{kinematical} \\ \text{params} \\ \text{in terms} \\ \text{of} \\ \text{integral} \\ \text{vars}}}{\cdots})^{1-ks_t}$$

then subtract
integrals at fixed
values of external momenta.

7.3 The renormalization group

For simplicity consider the single scale case
and expand the Green's functions

$$\boxed{G^r(x, L) = 1 - \mathrm{sgn}(s_r) \sum_k L^k \gamma_k^r(x)}$$

For a propagator e the RG eqn says

$$\Big(\frac{\partial}{\partial L} + \underset{\substack{\uparrow \\ \beta\text{-function}}}{\beta(x)} \frac{\partial}{\partial x} - 2\underset{\substack{\uparrow \\ \text{anomalous dim} \\ \text{of propagator e}}}{\gamma^e(x)}\Big) G^e(x, L) = 0$$

$$\text{so } \gamma^e = -\underset{\substack{\uparrow \\ \text{because } L = \log \frac{q^2}{\mu^2}}}{\frac{1}{2}} \frac{\partial}{\partial L} G^e(x, L)\Big|_{L=0} = \frac{1}{2}\gamma_1^e$$

$$\boxed{\text{so } \Big(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} - \gamma_1^e(x)\Big) G^e(x, L) = 0}$$

$$\text{and } \beta(\alpha) = \frac{\partial}{\partial L} \times \underset{\substack{\uparrow \\ \text{invariant} \\ \text{charge}}}{\underline{\phi(Q)}} \Big|_{L=0}$$

so comb invariant charge after applying Feynman rules

$$= \frac{\partial}{\partial L} \times \left(\frac{G^v(x, L)^2}{\prod_{e \text{ meets } v} G^e(x, L)}\right)^{\frac{1}{\mathrm{val}\,v - 2}}\Bigg|_{L=0} \quad \text{by Slavnov-Taylor}$$

$$= \frac{1 \cdot x}{\mathrm{val}\,v - 2}\Big(2\gamma_1^v + \sum_{\substack{e \text{ meets } v \\ \text{with mult}}} \gamma_1^e\Big)$$

$$\text{but also } \beta(x) = \frac{\partial}{\partial L} \times \phi(Q)\Big|_{L=0} = \frac{\partial}{\partial L} \times \prod_r G^r(x, L)^{-s_r}\Big|_{L=0} = \frac{\partial}{\partial L} \times (1 + |s_r|\gamma_1^r L + O(L^2))\Big|_{L=0}$$

$$= x \sum |s_r| \gamma_1^r$$

for vertices the RGE says

$$\left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} - \sum_{\substack{e \text{ fermion} \\ v}} \gamma^{e}(x)\right) \underbrace{x^{\frac{\text{val}(v)-2}{2}}} G^{v}(x,L) = 0$$

why? because normalized tree level to 1

but tree vertex has a contribution of the coupling.

So

$$0 = x^{\frac{\text{val}v-2}{2}} \frac{\partial}{\partial L} G^{v}(x,L) + \beta(x)\left(\frac{\text{val}v-2}{2}\right) x^{\frac{\text{val}v-4}{2}} G^{v}(x,L)$$
$$+ x^{\frac{\text{val}v-2}{2}} \beta(x)\frac{\partial}{\partial x} G^{v}(x,L) - x^{\frac{\text{val}v-2}{2}} \frac{1}{2}\sum_{e\ni v}\gamma_1^{e}(x) G^{v}(x,L)$$
$$= x^{\frac{\text{val}v-2}{2}} \left(\frac{\partial}{\partial L} G^{v}(x,L) + \beta(x)\frac{\partial}{\partial x} G^{v}(x,L) + \left(\left(\frac{\text{val}v-2}{2}\right)\left(\frac{1}{\text{val}v-2}\right)\left(2\gamma_1^{v} + \sum_{e\ni v}\gamma_1^{e}\right) - \frac{1}{2}\sum_{e\ni v}\gamma_1^{e}\right) G^{v}(x,L)\right)$$
$$= x^{\frac{\text{val}v-2}{2}} \boxed{\left(\frac{\partial}{\partial L} G^{v}(x,L) + \beta(x)\frac{\partial}{\partial x} G^{v}(x,L) + \gamma_1^{v} G^{v}(x,L)\right)}$$

now cancel the $x^{\frac{\text{val}v-2}{2}}$

so for edges and vertices

$$\boxed{\left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} - \text{sgn}(s_r)\gamma_1^{r}(x)\right) G^{r}(x,L) = 0}$$

Now extract the coeff of L^{k-1} in this expression to get

$$k\gamma_k^{r}(x) + \sum_j |s_j| \gamma_1^{j}(x) \frac{\partial}{\partial x}\gamma_{k-1}^{r}(x) - \text{sgn}(s_r)\gamma_1^{r}(x)\gamma_{k-1}^{r} = 0$$

so $\boxed{\gamma_k^{r} = \frac{1}{k}\left(\text{sgn}(s_r)\gamma_1^{r} - \sum_j |s_j| \gamma_1^{j}\frac{\partial}{\partial x}\right)\gamma_{k-1}^{r}}$

This gives γ_k^{r} recursively in terms of lower γ_i^{j}

In the single eqn case this is less messy and says

$$\boxed{\gamma_k = \frac{1}{k}\gamma_1\left(\text{sgn}(s) - |s| x\frac{\partial}{\partial x}\right)\gamma_{k-1}}$$

we'll return to this Monday and Tuesday.

7.4 Why and refs

Finally some physical objects

This is the Connes-Kreimer language of renormalization hep-th/9912092

This approach to DSEs comes from Broadhurst Kreimer hep-th/0012146

and was developped in gen in arXiv 0810.2249

⑧ The Yukawa example

8.1 The differential rearrangement

- We had last time

$$G(\alpha, L) = 1 - \frac{\alpha}{q^2}\int d^4k \frac{k\cdot q}{k^2 G(\alpha, \log\frac{k^2}{\mu^2})(k+q)^2} - \cdots \Big|_{q^2=\mu^2}$$

$$L = \log\frac{q^2}{\mu^2}$$

- Use the Ansatz $G(\alpha, L) = 1 - \sum_{k\geq 1}\gamma_k(\alpha)L^k$

plug it in and get

$$\sum_{k\geq 1}\gamma_k(\alpha)L^k = \frac{\alpha}{q^2}\int d^4k \frac{k\cdot q}{k^2\left(1-\sum_{k\geq 1}\left(\gamma_k(\alpha)\left(\log\frac{k^2}{\mu^2}\right)^k\right)\right)(k+q)^2} - \cdots \Big|_{q^2=\mu^2}$$

$$= \frac{\alpha}{q^2}\int d^4k \sum_{l_1+\cdots+l_s=l} \frac{k\cdot q\, \gamma_{l_1}(\alpha)\cdots\gamma_{l_s}(\alpha)\log^l\left(\frac{k^2}{\mu^2}\right)}{k^2(k+q)^2} - \cdots \Big|_{q^2=\mu^2}$$

$$= \frac{\alpha}{q^2}\sum_{l_1+\cdots+l_s=l}\gamma_{l_1}(\alpha)\cdots\gamma_{l_s}(\alpha)\int d^4k \frac{\log^l\left(\frac{k^2}{\mu^2}\right)}{k^2(k+q)^2} - \cdots \Big|_{q^2=\mu^2}$$

now use

$$\frac{d^k}{d(-\rho)^k}y^{-\rho}\Big|_{\rho=0} = \log^k(y)$$

$$= \frac{\alpha}{q^2}\sum_{l_1+\cdots+l_s=l}\gamma_{l_1}(\alpha)\cdots\gamma_{l_s}(\alpha)\int d^4k \frac{k\cdot q\,\frac{\partial^l}{\partial(-\rho)^l}\left(\frac{k^2}{\mu^2}\right)^{-\rho}\Big|_{\rho=0}}{k^2(k+q)^2} - \cdots \Big|_{q^2=\mu^2}$$

$$= \frac{\alpha}{q^2}\sum_{l_1+\cdots+l_s=l}\gamma_{l_1}(\alpha)\cdots\gamma_{l_s}(\alpha)\frac{\partial^l}{\partial(-\rho)^l}\left((\mu^2)^\rho\int d^4k \frac{k\cdot q}{(k^2)^{1+\rho}(k+q)^2} - \cdots \Big|_{q^2=\mu^2}\right)\Bigg|_{\rho=0}$$

↑ note analytic reg occurs naturally

$$= \alpha\left(1-\sum_{k\geq 1}\gamma_k(\alpha)\frac{\partial^k}{\partial(-\rho)^k}\right)^{-1}\left(\frac{(\mu^2)^\rho}{(q^2)^\rho}\int d^4k \frac{k_0\cdot q_0}{(k_0^2)^{1+\rho}(k_0+q_0)^2} - \cdots \Big|_{q^2=\mu^2}\right)\Bigg|_{\rho=0}$$

where $q = rq_0$ $r\in\mathbb{R}$

$r^2=q^2$ $q_0^2=1$ $k=rk_0$

so see the scaling clearly

$$= \alpha\, G\left(\alpha, \frac{\partial}{\partial(-\rho)}\right)^{-1} \left(\left(\left(\frac{\mu^2}{q^2}\right)^{\rho} - 1\right) \underbrace{\int d^4k \frac{k \cdot \xi}{(k^2)^{1+\rho}(k+\xi)^2}\Bigg|_{\xi^2=1}}_{} \right) \Bigg|_{\rho=0}$$

F(ρ) regularized integral
for the primitive
(regularize only at the insertion
place)

$$= \alpha\, G\left(\alpha, \frac{\partial}{\partial(-\rho)}\right)^{-1} \left(e^{-L\rho} - 1\right) F(\rho)\Big|_{\rho=0}$$

so $$\boxed{G(\alpha, L) = 1 - \alpha\, G\left(\alpha, \frac{\partial}{\partial(-\rho)}\right)^{-1} \left(e^{-L\rho} - 1\right) F(\rho)\Big|_{\rho=0}} \qquad (*)$$

8.2 The differential equation

Now it turns out $F(\rho) = \frac{-1}{\rho(2-\rho)}$ (separate into parallel and orthogonal parts and just do it)

Consider the coefficients of L and L^2 on each side of $(*)$ (only places L appears)

for L: $\gamma_1(\alpha) = -\alpha\, G\left(\alpha, \frac{\partial}{\partial(-\rho)}\right)^{-1} \rho F(\rho)\Big|_{\rho=0}$ $\left(e^{-L\rho} = 1 - L\rho + \frac{L^2\rho^2}{2} - \dots\right)$

for L^2: $\gamma_2(\alpha) = \alpha\, G\left(\alpha, \frac{\partial}{\partial(-\rho)}\right)^{-1} \frac{\rho^2}{2} F(\rho)\Big|_{\rho=0}$

But $F(\rho) = \frac{-1}{\rho(2-\rho)}$ so $\rho^2 F(\rho) = \frac{-\rho}{2-\rho} = \frac{-\frac{\rho}{2}}{1-\frac{\rho}{2}} = -\left(\frac{\rho}{2} + \frac{\rho^2}{4} + \frac{\rho^3}{8} + \dots\right)$

$= 1 - \left(1 + \frac{\rho}{2} + \frac{\rho^2}{4} + \frac{\rho^3}{3} + \dots\right)$

$= 1 - \rho F(\rho)$

so $$2\gamma_2(\alpha) = \alpha\, G\left(\alpha, \frac{\partial}{\partial(-\rho)}\right)^{-1} \left(1 + \rho F(\rho)\right)\Big|_{\rho=0}$$

$$= \alpha\, G\left(\alpha, \frac{\partial}{\partial(-\rho)}\right)^{-1} 1 - \gamma_1(\alpha)$$

$$= \alpha - \gamma_1(\alpha)$$

so $$\boxed{\gamma_1(\alpha) = \alpha - 2\gamma_2(\alpha)}$$

Furthermore recall the renormalization group calculation from last time

$$\left(\frac{\partial}{\partial L} + \beta(\alpha)\frac{\partial}{\partial \alpha} - 2\gamma(\alpha)\right)G(\alpha, L) = 0$$

where $\gamma(\alpha) = -\frac{1}{2}\frac{\partial}{\partial L}G(\alpha, L)\Big|_{L=0} = \frac{1}{2}\gamma_1(\alpha)$ the anomalous dimension

and $\beta(\alpha) = \frac{\partial}{\partial L}\alpha\,\phi(Q)$ the β-function

but just for this model, not all of Yukawa theory

so $\phi Q = G(\alpha, L)^{-2}$.

since $X = \mathbb{1} - xB_+\left(\frac{1}{X}\right)$

$= \mathbb{1} - xB_+\left(\frac{X}{X^2}\right)$

$= \mathbb{1} - xB_+(XQ)$

where $Q = X^{-2}$

so $\beta(\alpha) = 2\alpha\gamma_1(\alpha)$

so $$\left(\frac{\partial}{\partial L} + 2\alpha\gamma_1(\alpha)\frac{\partial}{\partial\alpha} - \gamma_1(\alpha)\right)G(\alpha, L) = 0$$

taking the coeff of L

$$-2\gamma_2(\alpha) - 2\gamma_1(\alpha)\,\alpha\frac{\partial}{\partial\alpha}\gamma_1(\alpha) + \gamma_1(\alpha)\gamma_1(\alpha)$$

so $\boxed{\gamma_2(\alpha) = \frac{1}{2}\gamma_1(\alpha)\left(1 - 2\alpha\frac{\partial}{\partial\alpha}\right)\gamma_1(\alpha)}$

so $\boxed{\gamma_1(\alpha) = \alpha - \gamma_1(\alpha)\left(1 - 2\alpha\frac{\partial}{\partial\alpha}\right)\gamma_1(\alpha).}$

so we have a differential equation for the anomalous dimension.

Now by cleverness (Broadhurst - Kreimer) or by Maple we can solve this getting an implicit solution

$$\exp\left(\frac{1}{2}\frac{(1+\gamma_1(\alpha))^2}{\alpha}\right)\sqrt{-\alpha} + \frac{1}{2}\operatorname{erf}\left(\frac{1}{\sqrt{2}}\frac{(1-\gamma_1(\alpha))}{\sqrt{\alpha}}\right)\sqrt{2\pi} = 0$$

And lets look at a picture of this

USCtalk3.xoj

To explain the picture better rewrite

$$\frac{\partial \gamma_1}{\partial \alpha} = \frac{\gamma_1 - \alpha + \gamma_1^2}{2\alpha\gamma_1} \leftarrow$$

but as we see in the picture there are crazy problems at the axes → problem is when signs change then direction of flow should also change

really we want $\frac{\partial \gamma_1}{\partial L}$ but $\frac{d\alpha}{\partial L} = \underset{\substack{\uparrow \\ \text{by def} \\ \text{of } \beta\text{-function}}}{\beta(\alpha)} = 2\alpha\gamma_1$

change in energy scale ie RG flow

so actually $\frac{\partial \gamma_1}{\partial L} = \gamma_1 - \alpha + \gamma_1^2$ $\frac{\partial \alpha}{\partial L} = 2\alpha\gamma_1$

This gives the second picture

8.3 Why and refs

Finally a <u>nonperturbative</u> result, albeit a special case

ref Broadhurst-Kreimer hep-th/0012146

⑨ Rewriting the DSEs

9.1 A different form of the DSE

In the Yukawa example we went from

$$G(\alpha, L) = 1 - \frac{\alpha}{q^2}\int d^4k \frac{k\cdot q}{k^2 G(\alpha, \log\frac{k^2}{\mu^2})(k+q)^2} - \cdots \Big|_{q^2=\mu^2}$$

to $G(\alpha, L) = 1 - \alpha G(\alpha, \frac{\partial}{\partial(-\rho)})^{-1}(e^{-L\rho} - 1)F(\rho)\Big|_{\rho=0}$

by
- Ansatz $G(x,L) = 1 - \sum_{k\geq 1} \gamma_k(x) L^k$
- $\frac{\partial^k}{\partial(-\rho)^k} y^{-\rho} \Big|_{\rho=0} = \log^k(y)$
- switch $\int, \sum$, then switch $\int, \frac{\partial}{\partial(-\rho)}$

analytic reg of the prim, hence $F(\rho)$ occurred naturally

We can do this for any reasonable example, but rather than establishing precise analytic conditions on when it is possible lets see how we go directly from the comb DSE to the differential form of the analytic DSE

For the Yukawa eg we had

$$X = \mathbb{1} - x B_+\left(\frac{1}{X}\right) \qquad Q = \frac{1}{X^2}$$

which became

$$G(x,L) = 1 - x\, G\left(x, \frac{\partial}{\partial(-\rho)}\right)^{-1} (e^{-L\rho} - 1) F(\rho) \Big|_{\rho=0}$$

In general we have the comb DSE

$$X^r = \mathbb{1} - \mathrm{sgn}(s_r) \sum_k x^k B_+^{k,r}(\underbrace{X^r Q^k}) \qquad Q = \prod_r (X^r)^{-s_r}$$

$F_{k,r}(\rho_1, \ldots, \rho_{n_{k,r}})$

each term appearing becomes

$$G^t\left(x, \cdots \frac{\partial}{\partial(-\rho_k)} \cdots\right)^{-\mathrm{sgn}(s_t)}$$

edges involved in the insertion.

get a total of $-ks_t$ G^t terms for $t \neq r$

get $1-ks_r$ G^r terms

and $e^{-L\rho}-1$ becomes $e^{-L(\rho_1+\dots+\rho_{n_{k,r}})}-1$

To make this a little less hairy consider the special case of one insertion place. Get

$$G^r(x,L) = 1 - \mathrm{sgn}(s_r)\sum_k x^k \left(G^r\left(x,\frac{\partial}{\partial(-\rho)}\right)^{1-s_r k}\prod_{j\neq r} G^j\left(x,\frac{\partial}{\partial(-\rho)}\right)^{-s_j k}\right)$$

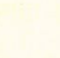
$$(e^{-L\rho}-1)\,F_{k,r}(\rho)\Big|_{\rho=0}$$

9.2 Symmetric insertion

Of course one insertion place is not enough (certainly don't get all graphs and furthermore can get really different kinds of answers → different transcendental #s)

How do we fix this?

Order by order the error when inserting in one place rather than in all is primitive – thus it fits into the B_+ framework hence we can still get the DSEs and so still have nonperturbative info

eg consider $X = 1 - xB_+^{\frac{1}{2}}$  $\left(\frac{1}{X^2}\right)$ in ϕ^3

$X = \mathbb{1} - x\frac{1}{2}$  $- x^2\frac{1}{2}$  $- x^3\Big(\frac{1}{8}$  $+\frac{1}{2}$  $+\frac{1}{4}$  $\Big)$

$- \dots$

(p33)

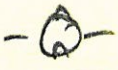
consider instead $X = \mathbb{1} - x B_+^{\frac{1}{2}}$  $\left(\frac{1}{X^2}\right)$

insert only here

still X^2 not X

$X = \mathbb{1} - \frac{x}{2}$  $- \frac{x^2}{2}$  $- x^3\Big(\frac{1}{2}$  $+ \frac{3}{8}$  $\Big)$

$- \ldots$

so difference is $\frac{1}{8}$  $- \frac{1}{8}$ 

call this q_3

check $\Delta(q_3) = q_3 \otimes \mathbb{1} + \mathbb{1} \otimes q_3 + \frac{1}{4}$  $+ \frac{1}{8}$ 

$- \frac{1}{4}$  $- \frac{1}{8}$ 

$= q_3 \otimes \mathbb{1} + \mathbb{1} \otimes q_3$

so consider $X = \mathbb{1} - x B_+^{\frac{1}{2}}$  $(XQ) - x^3 B_+^{q_3}(XQ^3) \qquad Q = X^{-3}$

and continue likewise.

In general the objects we need won't always have nice pictoral reps (eg inserting twice into the same vertex) so go back to tree reps to do this more generally

Choose a symmetric insertion scheme for the single insertion place so that the mellin transform $F_\gamma(\rho)$ has all loop variables involved in the insertion regularized equally by ρ and scales correctly.

Upshot is our analytic DSEs look like

$$G^r(x,L) = 1 - \mathrm{sgn}(s_r) \sum_k x^k \left(G^r\left(x, \frac{\partial}{\partial(-\rho)}\right)\right)^{1-s_r k} \prod_{j \neq r} G^j\left(x, \frac{\partial}{\partial(-\rho)}\right)^{-s_j k} (e^{-L\rho}-1)F(\rho)\Big|_{\rho=0}$$

in the single eqn case this reads

$$G(x,L) = 1 - \mathrm{sgn}(s) \sum_k x^k \, G\left(x, \frac{\partial}{\partial(-\rho)}\right)^{1-sk} (e^{-L\rho}-1)F(\rho)\Big|_{\rho=0} \quad (*)$$

9.3 Recurrences and P

Saturday we used the renormalization group to derive

$$\gamma^r_k = \frac{1}{k}\left(\mathrm{sgn}(s_r)\gamma^r_1 - \sum_j |s_j| \gamma^j_1 \frac{\partial}{\partial x} \right) \gamma^r_{k-1}$$

where $G^r(x,L) = 1 - \mathrm{sgn}(r) \sum_k L^k \gamma^r_k(x)$

or in the single equation case

$$\gamma_k = \frac{1}{k} \gamma_1 \left(\mathrm{sgn}(s) - |s| \, x \frac{\partial}{\partial x} \right) \gamma_{k-1}$$

These determine the γ_k $k > 1$ in terms of γ_1

Now consider the DSE $(*)$ I will write the single eqn case for now, but the system case is only messier not harder

Take the coeff of L

$$(**) \quad -\mathrm{sgn}(s)\gamma_1(x) = -\mathrm{sgn}(s) \sum_k x^k G\left(x, \frac{\partial}{\partial(-\rho)}\right)^{1-sk} (-\rho F(\rho))\Big|_{\rho=0}$$

Take the coeff of L^2

$$(***) \quad -\mathrm{sgn}(s)\gamma_2(x) = -\mathrm{sgn}(s) \sum_k x^k G\left(x, \frac{\partial}{\partial(-\rho)}\right)^{1-sk} \left(\frac{\rho^2}{2} F(\rho)\right)\Big|_{\rho=0}$$

In the Yukawa case $F(\rho)$ was a geometric series

so $-\rho F(\rho)$ & $\frac{\rho^2}{2} F(\rho)$ were related: $\rho^2 F(\rho) = 1 - \rho F(\rho)$

so write each $F_k(\rho) = \underset{\substack{\uparrow \\ \text{good part}}}{\frac{r_k}{\rho(1-\rho)}} + \text{rest}$

Furthermore we can replace the extra powers of ρ in the rest by powers of L (in some sense RG again)

specifically $x^k G(x, \frac{\partial}{\partial(-\rho)})^{1-sk} \rho^\ell \big|_{\rho=0}$

has no term of degree less than $k+\ell$ in x

since $\gamma_i(x)$ begins with an x^i term

so $$x G(x, \frac{\partial}{\partial(-\rho)})^{1-sk} (e^{-L\rho} - 1) \frac{1}{\rho(1-\rho)}\bigg|_{\rho=0} = -Lx^k + O(x^{k+1})$$

so trading terms between primitives as necessary in place of $F_k(\rho)$ we have $\frac{r_k}{\rho(1-\rho)} + \sum_{1 \le i \le k} \frac{r_{k,i} L^i}{\rho}$

so (**) gives $$\gamma_1(x) = -\sum_k x^k G(x, \frac{\partial}{\partial(-\rho)})^{1-sk} \left(\frac{r_k}{1-\rho}\right)\bigg|_{\rho=0}$$

$$2\gamma_2(x) = \sum_k x^k G(x, \frac{\partial}{\partial(-\rho)})^{1-sk} \left(\underbrace{\frac{\rho r_k}{1-\rho}}_{\substack{\parallel \\ \frac{r_k}{1-\rho} - r_k}} - r_{k,1}\right)\bigg|_{\rho=0}$$

so $$\gamma_1 + 2\gamma_2 = \underbrace{-\sum (r_k + 2r_{k,1}) x^k}_{P(x)}$$

with a little more work can show

the $r_{k,1} = 0$ so $P(x) = \sum(-r_k)x^k$

is just the leading part of the primitive

Finally $\gamma_1 = -2\gamma_2 + P(x)$

$$= \gamma_1(-\mathrm{sgn}(s) + |s|\, x\frac{\partial}{\partial x})\gamma_1 + P(x)$$

So we have a DE for γ_1 as in the Yukawa case

9.4 why and note

The DE tells us the nonperturbative behavior

cf arXiv: 0810.2249

⑩ Pictures of the DE

10.1 playing with $P>0$ and $s>0$

The DE is $\gamma_1 = \gamma_1(-1 + s x\frac{\partial}{\partial x})\gamma_1 + P$

so $\frac{d\gamma_1}{dx} = \frac{\gamma_1 - P + \gamma_1^2}{s x \gamma_1}$ or $\begin{cases} \frac{d\gamma_1}{dL} = \gamma_1 - P + \gamma_1^2 \\ \frac{dx}{dL} = s x \gamma_1 \end{cases}$

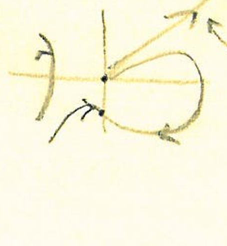
So s doesn't change the qualitative picture much

- show different behaviours

DRAW IN THE NULLCLINE

nullcline is $\frac{d\gamma_1}{dx} = 0$

- show animation



question do these exist

(answer coming up)

10.2 QED

$P>0$ $s>0$ is the situation for QED
for $x>0$

$P>0$ in QED because we can calculate perturbatively
↑ so know behaviour near 0
at lest
near 0

QED is a single equation case — just the photon propagator
if you work in Baker, Johnson, Willey gauge
(a recursive choice of gauge which leaves only
the photon needing renormalization)

$s=1$ for the photon propagator

as  has no insertion place for a
photon so B_+ has argument $\mathbb{1}$

$$\mathbb{1} = \frac{X}{\textcircled{X}} \leftarrow Q = X^{-1} \quad \text{so } s=1$$

looks pretty similar (show pictures)
unless you get a zero for P
in which case you also get a 0 for γ_1, hence β
but this not expected to occur

Results (van Baalen, Kreimer, Uminsky, I)

let $s>0$, P C^2 $\sim >0$ for $x>0$
Then there exist solutions which are global in x iff

$$(*) \quad \int_{x_0}^{\infty} \frac{P(z)}{z^{1+\frac{3}{s}}}\,dz < \infty \quad \text{for some } x_0>0$$

Note

- for $P(x) = x$ the test is $\int_{Y_0}^{\infty} \frac{1}{z^{\frac{2}{s}}} dz < \infty$

  So we have global (in x) solutions iff $s<2$

- for QED $P(x)$ can grow at most $o(x^2)$ to get global solns
- If $P(x) \xrightarrow[x \to \infty]{} C < \infty$ then for any s get global solns

If P satisfies $(*)$ then there is a unique separatrix

all solutions above it exist for all x
all solutions below it hit the x axis and turn back around to -1

What about growth in L?

Assume further P is increasing and satisfies $(*)$

then every solution γ_1 above the separatrix grows like

$$C_1 x^{\frac{1}{s}} \le \gamma_1 \le C_2 x^{\frac{1}{s}} \quad \text{for} \quad x \to \infty$$

These solutions must go to ∞ in finite L (Landau poles)

the separatrix $\gamma_1^* \le \min \lim_{x\to\infty} \left\{ \underset{\uparrow \atop \text{the nullcline}}{\frac{\sqrt{1+4P}-1}{2}}, C x^{\frac{1}{s}} \right\}$

so if $P \to C < \infty$

then γ_1^* is bdd but all other global solns are not

so γ_1^* MIGHT NOT BE A LANDAU POLE

Precisely γ_1^* is a Landau pole iff

$$\int_{x_0}^{\infty} \frac{2dz}{\underbrace{z\left(\sqrt{1+4P(z)}-1\right)}_{\text{null cline again}}} < \infty$$

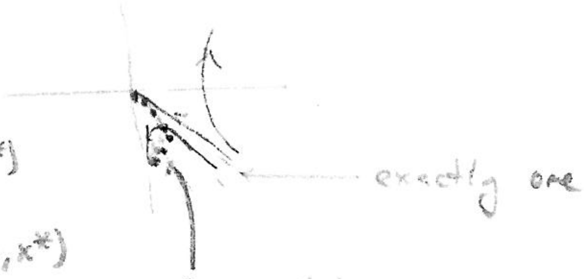
(p39)

10.3 QCD

Suppose $P<0$ for small $x>0$. This really changes the picture (show picture)

Note the behaviours

hypotheses
$$\begin{cases} P(0)=0 \\ P(x)<0 \\ \quad \text{for } x\in(0,x^*) \end{cases}$$
$P'(x)<0$
for $x\in(0,x^*)$



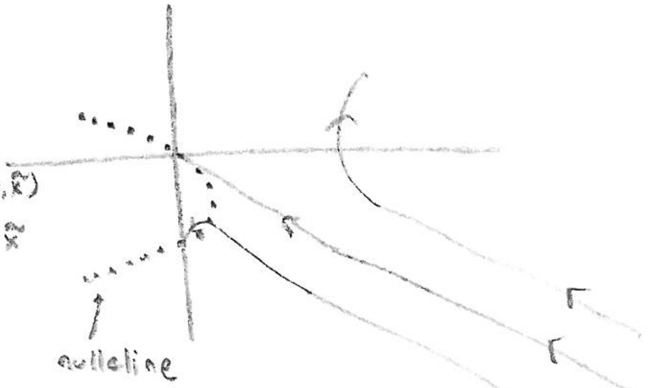
show nullcline
(doesnt necess join
(depends if P hits $-\frac{1}{4}$ or not)

The solution which is 0 at $x=0$ is the one we want as $L\to\infty$ it tends to 0 so it is asymptotically free — there is only one asymptotically free soln.

This is the situation for massless QCD in background field gauge
(mass would add another equation to get a system)

If P hits $-\frac{1}{4}$ looks like this

hypotheses
$\exists\,\tilde{x}$ st $P(\tilde{x})=-\frac{1}{4}$
$-\frac{1}{4}<P(x)<0$ $x\in(0,\tilde{x})$
$P(x)<-\frac{1}{4}$ $x>\tilde{x}$

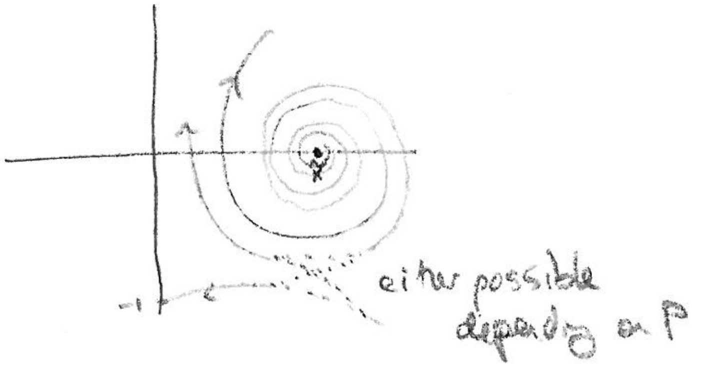


for these solutions
$x\to\infty$
$L\to c<\infty$

$\gamma_1(x)\sim -cx-1$ if $P(x)\lesssim x$
$\gamma_1(x)\sim -cx-d$ if $x\lesssim P(x)\lesssim x^2$
$\gamma_1(x)\gtrsim x$ otherwise
(depends on growth of P roughly $\sqrt{P}$)

- no possibility of slow growth
- a <u>region</u> of linear growth.

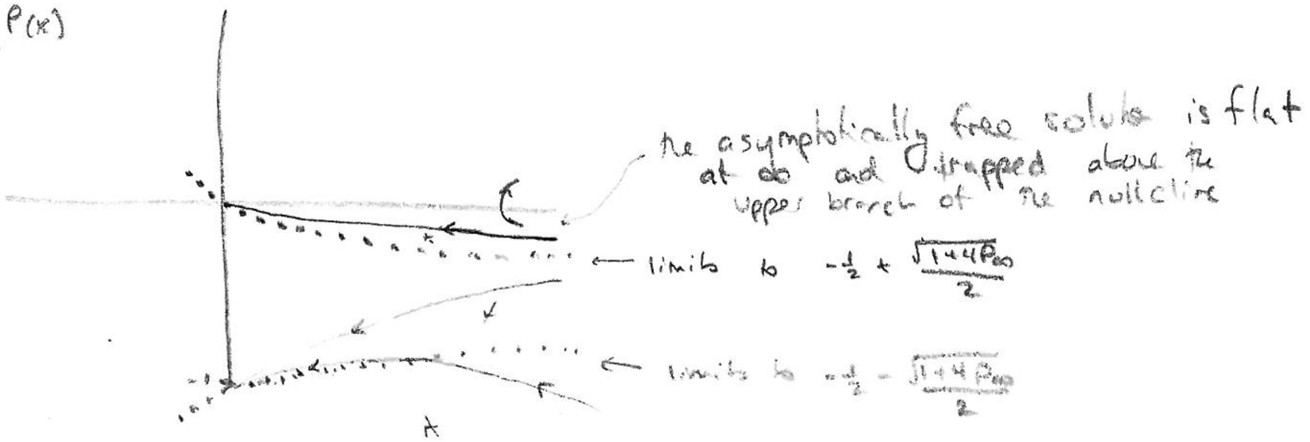
If $\exists\, \tilde{x}$ wih $P(\tilde{x}) = 0$ it looks like this



- very diffrent from when P=0 in QED
- spirals give ∞'ly many diff'nt 0s of β-fn → not physical
- asymptotically free soln may or may not spiral depending on P.

If $P(x)$ stays above $-\frac{1}{4}$ it looks like this

Let $P_\infty = \lim_{x\to\infty} P(x)$



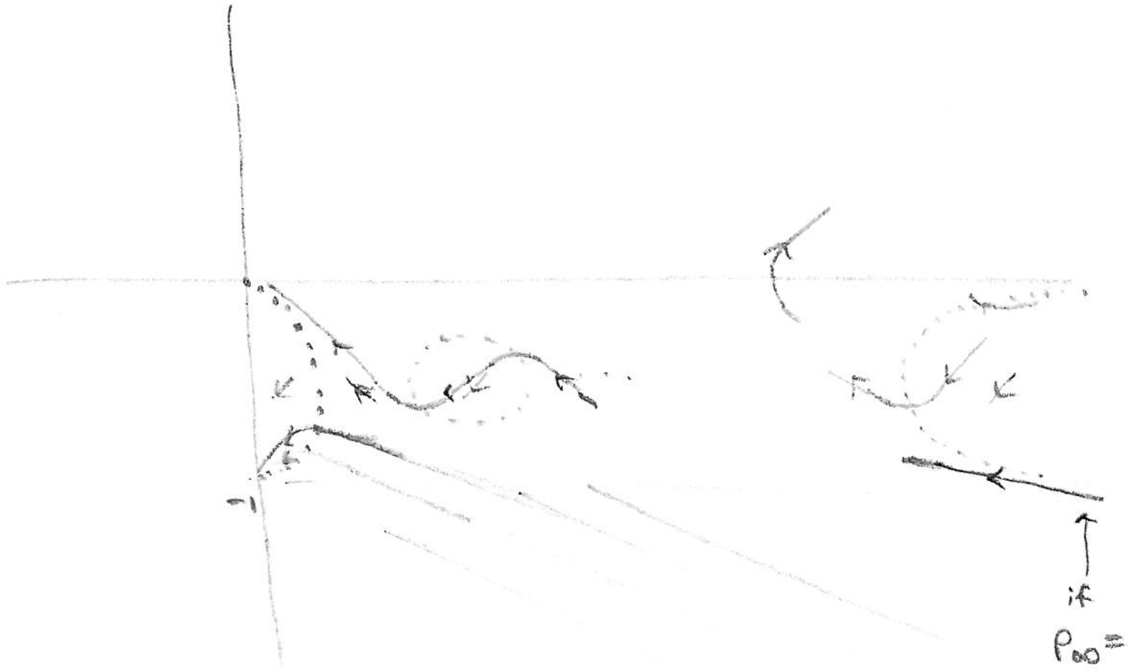
Now it would be nice to see <u>confinement</u> somehow
what would that look like in this language?
Answer: $\gamma_1 \to -1$ as $x \to -\infty$

Is that possible?
Yes but not in a truly satisfactory way

if $\exists \tilde{x}$ $P(\tilde{x}) = -\frac{1}{4}$

w/ $\lim_{x \to \infty} P(x) = P_\infty > -\frac{1}{4}$

then looks like



if $P_\infty = 0$ this soln goes to -1

why should the asymptotically free and the confining solution match → no idea

perhaps some sort of symmetry (a "rotation" in the picture) forces it or perhaps this set up is not rich enough to see this aspect of reality.

10.4 Why, refs

That was the punchline

refs vanBaalen, Kreimer, Uminsky, Y Ann Phys 234, 1 (2009) 205-219

and Ann Phys 325, 2, (210) 300-324

(arXiv: 0805.0826 and 0906.1754)

⑪ A chord diagram expansion

11.1 Rooted connected chord diagrams

Some of that was pretty adhoc so lets go back to being perturbative and in one special case try to better understand what's going on. To do that we need some more combinatorics first

Def rooted connected chord diagram

eg

Def intersection graph and terminal chords

eg

Def recursive chord order

eg

Note
- recursive chord order can be different from counterclockwise order
- terminal chords are the base case

Def $w(C) = \{ i , c_i$ is terminal$\}$ using recursive chord order

$i(C)$ list of differences of successive elts in $w(C)$ padded with 0s to $|C|-1$ elts

$$f_C = \prod_{i \in i(C)} f_i$$

$b(C)$ = min index of a terminal chord

eg

(p43)

11.2 Solving a DSE as a chord diagram expansion

Consider $$G(x,L) = 1 - xG\left(x, \frac{\partial}{\partial(-\rho)}\right)^{-1}(e^{-L\rho}-1)F(\rho)\Big|_{\rho=0}$$

expand $F(\rho) = \frac{f_0}{\rho} + f_1 + f_2\rho + f_3\rho^2 + \dots$

& use $G(x,L) = 1 - \sum_{n\geq 1}\gamma_n(x)L^n$

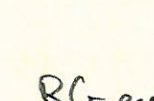
Then $$\gamma_i(x) = \frac{(-1)^i}{i!}\sum_{\substack{C\\ b(C)\geq i}} x^{|C|} f_C f_{b(C)-i}$$ (sum is our rooted connected chord diagrams with)

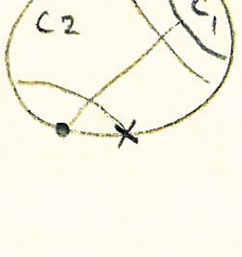
solves the DSE

How do we prove this? 2 recurrences both interesting in their own right

For the first recurrence:

Def root share decomposition

eg 



Recall the RG eqn gave

$$* \quad \boxed{\gamma_k = \frac{1}{k}\gamma_1\left(-1 + 2x\frac{d}{dx}\right)\gamma_{k-1}}$$

A classical recurrence (Stein, this proof Nijenhuis + Wilf) says

$$s_n = \sum_{k=1}^{n-1}(2k-1)s_k s_{n-k}$$ for $n \geq 2$

where s_n is the number of rooted connected chord diagrams with n chords

(p44)

Proof by root share decomp

in a chord diagram C_2 with k chords
have $2k-1$ places to put C_1

Keeping track of the terminal chords we get

$$** \quad \boxed{g_{k,i} = \sum_{\ell=1}^{i-1} (2\ell-1)\, g_{1,i-\ell}\, g_{k-1,\ell}} \qquad 2 \le k \le i$$

where $g_{k,i} = \sum_{\substack{C \\ |C|=i \\ b(C) \ge k}} f_C f_{b(C)-k}$

eg $g_{2,2}$:

$i(C) = (0)$
$b(C) = 2$ contribts f_0^2

g_{11} g_{11}

$i(C) = \emptyset$
$b(C) = 1$ each contributs f_0

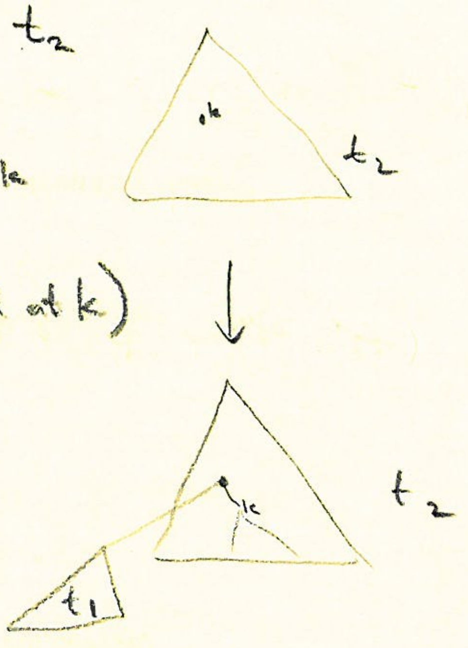
then with $\gamma_k = \frac{(-1)^k}{k!} \sum_{i \ge k} g_{k,i} x^i$ ** becoms *

Now we know γ_k correctly depends on lower γ_i
We only now need to check γ_1 is correct. This is the purpose of the second recurrence

For the second recurrence:

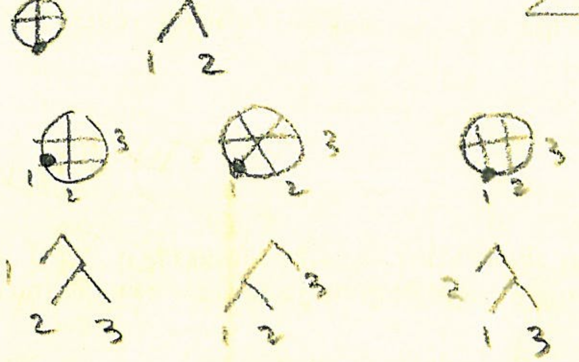
Def Binary tree decomp
Given a rooted connected chord diagram C
if $|C| = 1$ the tree is $\circ 1$
if $|C| > 1$ let C_1, C_2 be the root share decomp
say in k^{th} slot

Let t_1, t_2 be the trees for C_1, C_2
find the k^{th} vertex of t_2
insert t_1 there
(make a new vertex at k
wih right child t_1
left child tree rooted at k)



eg

so



To prove the result it remains to show the chord diagram γ_1 satisfies

$$\gamma_1 = x\Big(1-\sum_{k\geq 1}\gamma_k(\partial_{-\rho})^k\Big)^{-1}(-\rho)F(\rho)\Big|_{\rho=0}$$ (L^1 term from the DSE)

After a couple of pages of manipulations can show it suffices to show

$$\sum_{\substack{c\\|c|=i+1\\b(c)=j+1}} f_{i(c)} = \sum_{k=1}^{i}\sum_{l=1}^{j}\binom{j}{l}\Bigg(\sum_{\substack{c\\|c|=k\\b(c)\geq l}} f_{i(c)}f_{b(c)-l-1}\Bigg)\Bigg(\sum_{\substack{c\\|c|=i-k+1\\b(c)=j-l+1}} f_{i(c)}\Bigg)$$

This is exactly what decomposing the tree into its left and right subtrees does, which proves the chord diagram expansion.

11.3 Why and refs

- This chord diagram expansion removes all the recursion
- gives the Green's function as a sort of multivariate generating function
- Can say something about asymptotics of the series.
- Still lots of questions
  - explore the objects further eg.
    how to see the decomposition of the second recurrence at the level of chord diagrams rather than trees

    tricky eg

↑
not tree

for

which is

  - other values of s
    more general DSEs

ref this is joint work with Nicolas Marie
there will be a paper soon; for now see my talk from

www.mathematik.hu-berlin.de/
~maphy/confJune2012.html