Argonne NATIONAL LABORATORY

Emergence of DSEs in Real-World QCD

Published collaborations: 2010-present

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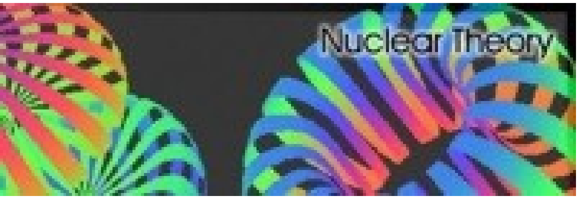

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Exploring Nuclear Matter - Quarks to Stars

Physics Division
www.phy.anl.gov/theory/staff/cdr.html

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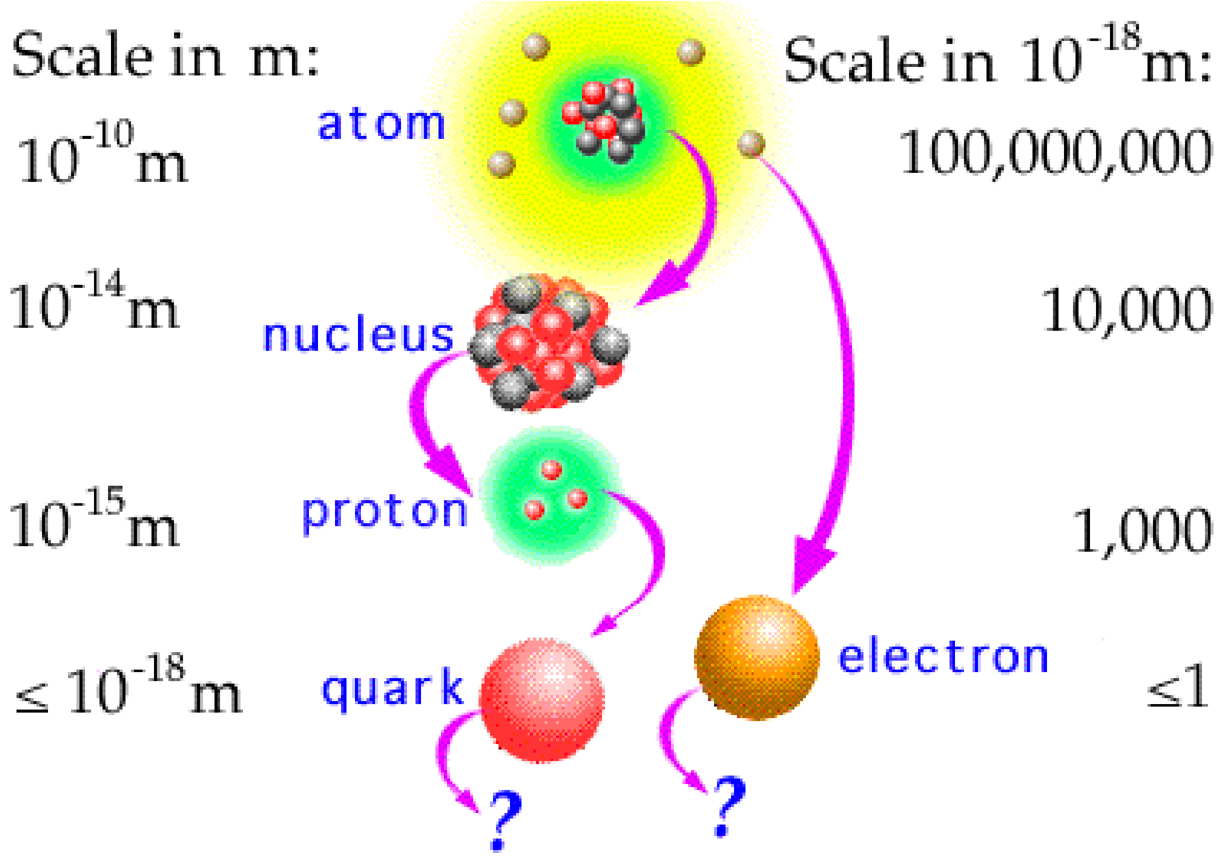
Recommended reading

- C. D. Roberts, *"Strong QCD and Dyson-Schwinger Equations,"* arXiv:1203.5341 [nucl-th]. Notes based on 5 lectures to the conference on "Dyson-Schwinger Equations & Faà di Bruno Hopf Algebras in Physics and Combinatorics (DSFdB2011)," Institut de Recherche Mathématique Avancée, l'Universite de Strasbourg et CNRS, Strasbourg, France, 27.06-01.07/2011. To appear in "IRMA Lectures in Mathematics & Theoretical Physics," published by the European Mathematical Society (EMS)
- C.D. Roberts, M.S. Bhagwat, A. Höll and S.V. Wright, "Aspects of Hadron Physics," Eur. Phys. J. Special Topics 140 (2007) pp. 53-116
- A. Höll, C.D. Roberts and S.V. Wright, nucl-th/0601071, "Hadron Physics and Dyson-Schwinger Equations" (103 pages)
- C.D. Roberts (2002): "Primer for Quantum Field Theory in Hadron Physics" (http://www.phy.anl.gov/theory/ztfr/LecNotes.pdf)
- C. D. Roberts and A. G. Williams,"Dyson-Schwinger equations and their application to hadronic physics," Prog. Part. Nucl. Phys. **33** (1994) 477

Recommended reading

- A. Bashir, Lei Chang, Ian C. Cloët, Bruno El-Bennich, Yu-xin Liu, Craig D. Roberts and Peter C. Tandy, "*Collective perspective on advances in Dyson-Schwinger Equation QCD,*" arXiv:1201.3366 [nucl-th], Commun. Theor. Phys. **58** (2012) pp. 79-134
- R.J. Holt and C.D. Roberts, "*Distribution Functions of the Nucleon and Pion in the Valence Region,*" arXiv:1002.4666 [nucl-th], Rev. Mod. Phys. **82** (2010) pp. 2991-3044
- C.D. Roberts , "*Hadron Properties and Dyson-Schwinger Equations,*" arXiv:0712.0633 [nucl-th], Prog. Part. Nucl. Phys. **61** (2008) pp. 50-65
- P. Maris and C. D. Roberts, "Dyson-Schwinger equations: A tool for hadron physics," Int. J. Mod. Phys. E **12**, 297 (2003)
- C. D. Roberts and S. M. Schmidt, "Dyson-Schwinger equations: Density, temperature and continuum strong QCD," Prog. Part. Nucl. Phys. **45** (2000) S1



Standard Model of Particle Physics

Standard Model - History

- In the early 20th Century, the only matter particles known to exist were the proton, neutron, and electron.
- With the advent of cosmic ray science and particle accelerators, numerous additional particles were discovered:
  - muon (1937), pion (1947), kaon (1947), Roper resonance (1963), …
- By the mid-1960s, it was apparent that not all the particles could be *fundamental*.
  - A new paradigm was necessary.
- Gell-Mann's and Zweig's constituent-quark theory (1964) was a critical step forward.
  - Gell-Mann, Nobel Prize 1969: *"for his contributions and discoveries concerning the classification of elementary particles and their interactions"*.

- Over the more than forty intervening years, the theory now called the *Standard Model of Particle Physics* has passed almost all tests.

Standard Model - The Pieces

- Electromagnetism
  - Quantum electrodynamics, 1946-1950
  - Feynman, Schwinger, Tomonaga
    - Nobel Prize (1965):

      "for their fundamental work in quantum electrodynamics, with deep-ploughing consequences for the physics of elementary particles".
- Weak interaction
  - Radioactive decays, parity-violating decays, electron-neutrino scattering
  - Glashow, Salam, Weinberg - 1963-1973
    - Nobel Prize (1979):

      "for their contributions to the theory of the unified weak and electromagnetic interaction between elementary particles, including, inter alia, the prediction of the weak neutral current".

Standard Model - The Pieces

- Strong interaction
  - Existence and composition of the vast bulk of visible matter in the Universe:
    - proton, neutron
    - the forces that form them and bind them to form nuclei
    - responsible for more than 98% of the visible matter in the Universe

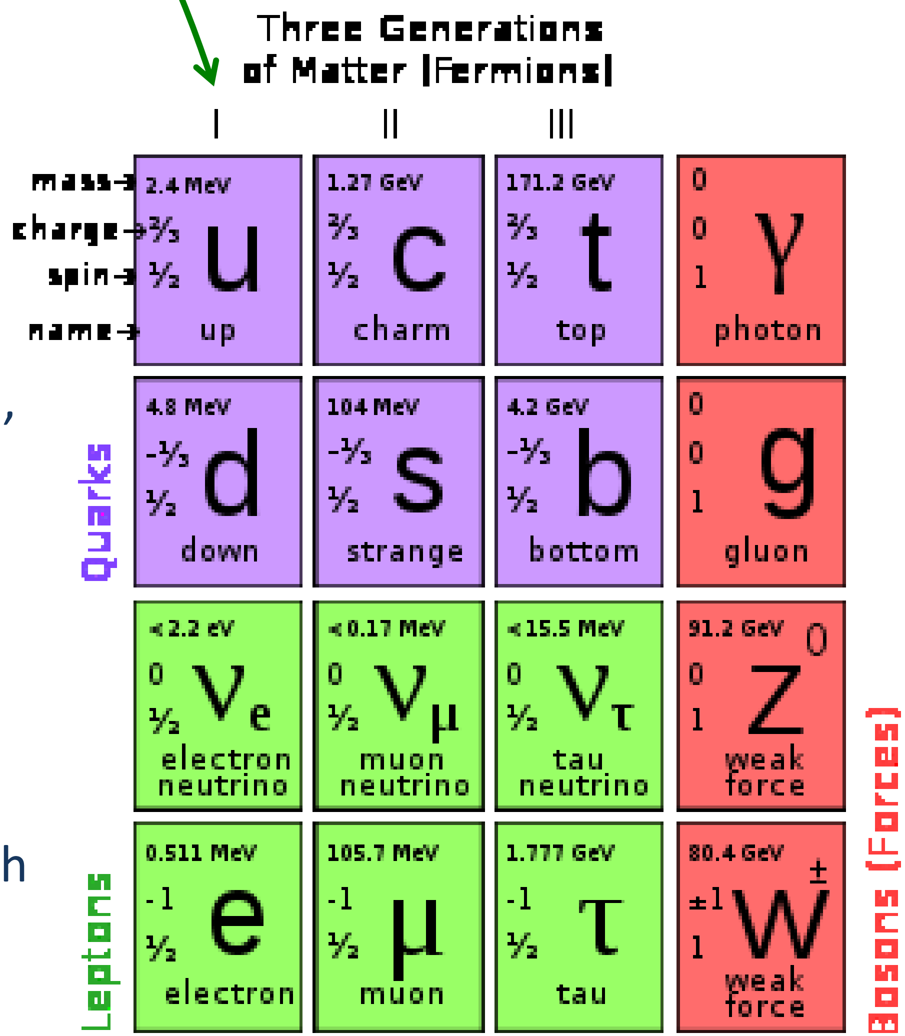
  - Politzer, Gross and Wilczek – 1973-1974
    Quantum Chromodynamics – QCD
    - Nobel Prize (2004):
      "for the discovery of asymptotic freedom in the theory of the strong interaction".
- NB.
  Worth noting that the nature of 95% of the matter in the Universe is completely unknown

Standard Model - Formulation

- The Standard Model of Particle Physics is a local gauge field theory, which can be completely expressed in a very compact form
- Lagrangian possesses $SU_c(3)xSU_L(2)xU_Y(1)$ gauge symmetry
  - 19 parameters, which must be determined through comparison with experiment
    - *Physics is an experimental science*
  - $SU_L(2)xU_Y(1)$ represents the electroweak theory
    - 17 of the parameters are here, most of them tied to the Higgs boson, the model's only fundamental scalar, which might now have been seen
    - This sector is essentially perturbative, so the parameters are readily determined
  - $SU_c(3)$ represents the strong interaction component
    - Just 2 of the parameters are intrinsic to $SU_c(3)$ – QCD
    - However, this is the really interesting sector because it is Nature's only example of a truly and essentially nonperturbative fundamental theory
    - Impact of the 2 parameters is not fully known

Standard Model - Formulation

- Known particle content of the Standard Model
- Discovery of the Higgs boson is one of the primary missions of the Large Hadron Collider
- LHC
  - Construction cost of $7 billion
  - Accelerate particles to almost the speed of light, in two parallel beams in a 27km tunnel 175m underground, before colliding them at interaction points
  - During a ten hour experiment , each beam will travel 10-billion km; i.e., almost 100-times the earth-sun distance
  - The energy of each collision will reach 14 TeV (*14 x 10*^{12} eV)
- Higgs *might* now have been found



$$\mathcal{L}_{\rm SM} = \mathcal{L}_{\rm Dirac} + \mathcal{L}_{\rm mass} + \mathcal{L}_{\rm gauge} + \mathcal{L}_{{\rm gauge}/\psi} \,. \qquad (1)$$

Here,

$$\mathcal{L}_{\rm Dirac} = i\bar{e}^i_{\rm L}\partial\!\!\!/ e^i_{\rm L} + i\bar{\nu}^i_{\rm L}\partial\!\!\!/ \nu^i_{\rm L} + i\bar{e}^i_{\rm R}\partial\!\!\!/ e^i_{\rm R} + i\bar{u}^i_{\rm L}\partial\!\!\!/ u^i_{\rm L} + i\bar{d}^i_{\rm L}\partial\!\!\!/ d^i_{\rm L} + i\bar{u}^i_{\rm R}\partial\!\!\!/ u^i_{\rm R} + i\bar{d}^i_{\rm R}\partial\!\!\!/ d^i_{\rm R} \,; \qquad (2)$$

$$\mathcal{L}_{\rm mass} = -v\left(\lambda^i_e \bar{e}^i_{\rm L} e^i_{\rm R} + \lambda^i_u \bar{u}^i_{\rm L} u^i_{\rm R} + \lambda^i_d \bar{d}^i_{\rm L} d^i_{\rm R} + {\rm h.c.}\right) - M_W^2 W^+_\mu W^{-\mu} - \frac{M_W^2}{2\cos^2\theta_W} Z_\mu Z^\mu \,; \qquad (3)$$

$$\mathcal{L}_{\rm gauge} = -\frac{1}{4}(G^a_{\mu\nu})^2 - \frac{1}{2}W^+_{\mu\nu}W^{-\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_{WZA} \,, \qquad (4)$$

where

$$\begin{aligned} G^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \boxed{g_3 f^{abc} A^b_\mu A^c_\nu} \\ W^\pm_{\mu\nu} &= \partial_\mu W^\pm_\nu - \partial_\nu W^\pm_\mu \\ Z_{\mu\nu} &= \partial_\mu Z_\nu - \partial_\nu Z_\mu \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \,, \end{aligned} \qquad (5)$$

and

$$\begin{aligned} \mathcal{L}_{WZA} = \; & i g_2 \cos\theta_W \left[\left(W^-_\mu W^+_\nu - W^-_\nu W^+_\mu\right)\partial^\mu Z^\nu + W^+_{\mu\nu}W^{-\mu}Z^\nu - W^-_{\mu\nu}W^{+\mu}Z^\nu\right] \\ & + ie\left[\left(W^-_\mu W^+_\nu - W^-_\nu W^+_\mu\right)\partial^\mu A^\nu + W^+_{\mu\nu}W^{-\mu}A^\nu - W^-_{\mu\nu}W^{+\mu}A^\nu\right] \\ & + g_2^2\cos^2\theta_W\left(W^+_\mu W^-_\nu Z^\mu Z^\nu - W^+_\mu W^{-\mu} Z_\nu Z^\nu\right) \\ & + g_2^2\left(W^+_\mu W^-_\nu A^\mu A^\nu - W^+_\mu W^{-\mu} A_\nu A^\nu\right) \\ & + g_2 e \cos\theta_W\left[W^+_\mu W^-_\nu\left(Z^\mu A^\nu + Z^\nu A^\mu\right) - 2W^+_\mu W^{-\mu} Z_\nu A^\nu\right] \\ & + \frac{1}{2}g_2^2\left(W^+_\mu W^-_\nu\right)\left(W^{+\mu}W^{-\nu} - W^{+\nu}W^{-\mu}\right) \,; \end{aligned} \qquad (6)$$

and

$$\mathcal{L}_{{\rm gauge}/\psi} = -g_3 A^a_\mu J^{\mu a}_{(3)} - g_2\left(W^+_\mu J^\mu_{W^+} + W^-_\mu J^\mu_{W^-} + Z_\mu J^\mu_Z\right) - e A_\mu J^\mu_A \,, \qquad (7)$$

where

$$\begin{aligned} J^{\mu a}_{(3)} &= \bar{u}^i\gamma^\mu T^a_{(3)} u^i + \bar{d}^i\gamma^\mu T^a_{(3)} d^i \\ J^\mu_{W^+} &= \frac{1}{\sqrt{2}}\left(\bar{\nu}^i_{\rm L}\gamma^\mu e^i_{\rm L} + V^{ij}\bar{u}^i_{\rm L}\gamma^\mu d^j_{\rm L}\right) \\ J^\mu_{W^-} &= (J^\mu_{W^+})^* \\ J^\mu_Z &= \frac{1}{\cos\theta_W}\left[\frac{1}{2}\bar{\nu}^i_{\rm L}\gamma^\mu\nu^i_{\rm L} + \left(-\frac{1}{2} + \sin^2\theta_W\right)\bar{e}^i_{\rm L}\gamma^\mu e^i_{\rm L} + (\sin^2\theta_W)\bar{e}^i_{\rm R}\gamma^\mu e^i_{\rm R}\right. \\ & \quad + \left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_W\right)\bar{u}^i_{\rm L}\gamma^\mu u^i_{\rm L} + \left(-\frac{2}{3}\sin^2\theta_W\right)\bar{u}^i_{\rm R}\gamma^\mu u^i_{\rm R} \\ & \quad \left. + \left(-\frac{1}{2} + \frac{1}{3}\sin^2\theta_W\right)\bar{d}^i_{\rm L}\gamma^\mu d^i_{\rm L} + \left(\frac{1}{3}\sin^2\theta_W\right)\bar{d}^i_{\rm R}\gamma^\mu d^i_{\rm R}\right] \\ J^\mu_A &= (-1)\bar{e}^i\gamma^\mu e^i + \left(\frac{2}{3}\right)\bar{u}^i\gamma^\mu u^i + \left(-\frac{1}{3}\right)\bar{d}^i\gamma^\mu d^i \,. \end{aligned} \qquad (8)$$

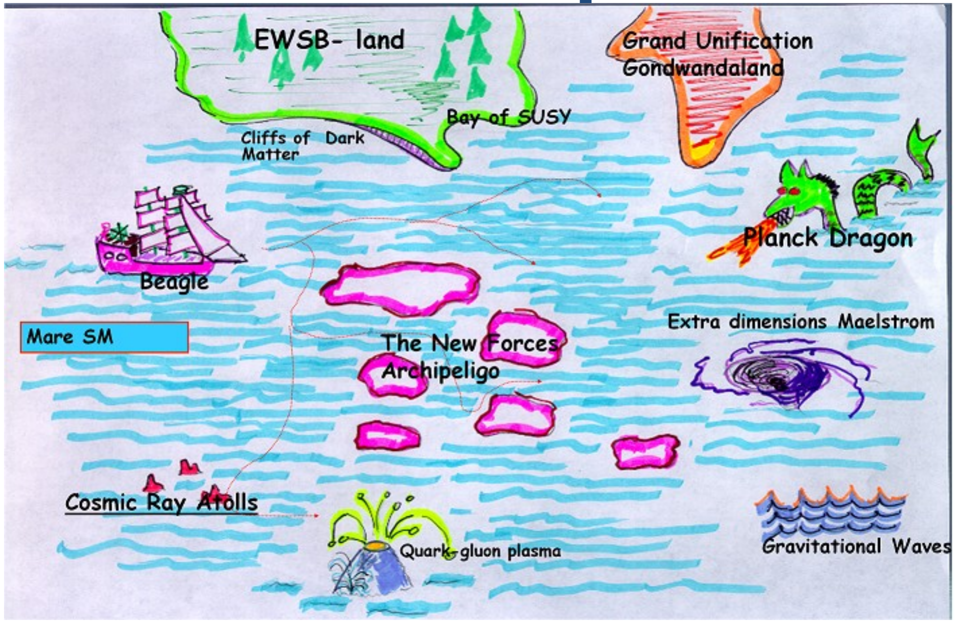
Standard Model - Formulation

- Very compact expression of the fundamental interactions that govern the composition of the bulk of known matter in the Universe
- This is the most important part; viz., gauge-boson self-interaction in QCD
  - Responsible for 98% of visible matter in the Universe
- QCD will be my primary focus

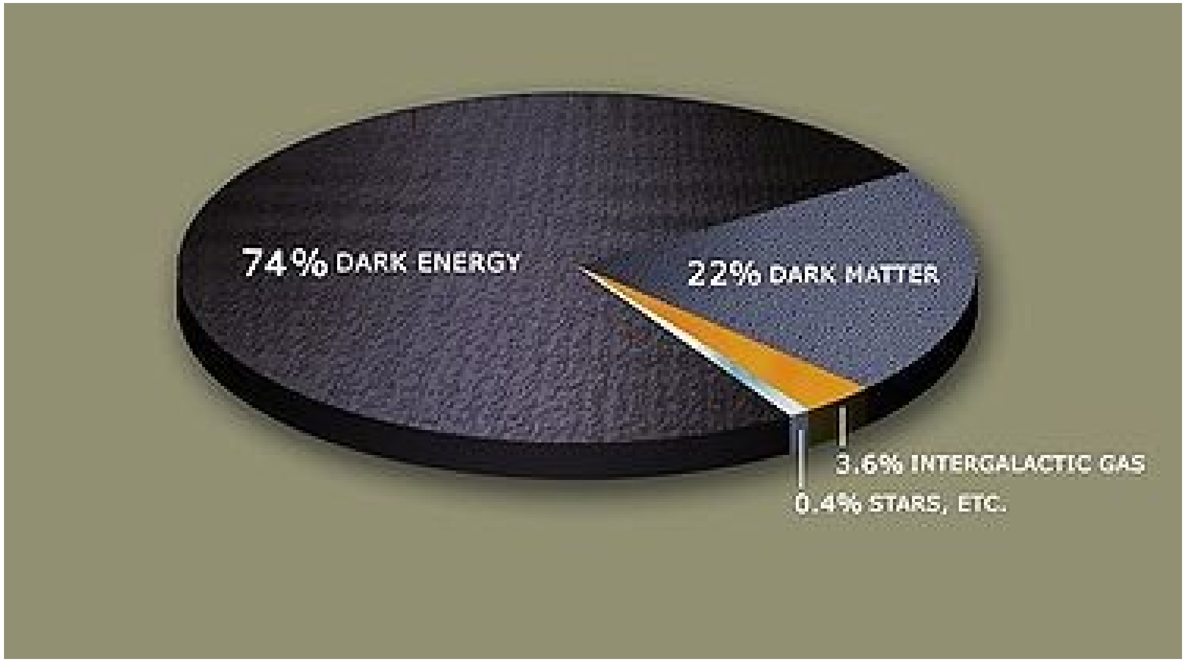
Standard Model - Complete?

- There are certainly phenomena Beyond the Standard Model
  - Neutrinos have mass, which is not true within the Standard Model
  - Empirical evidence: $\nu_e \leftrightarrow \nu_\mu, \nu_\tau$

    … neutrino flavour is not a constant of motion
    - The first experiment to detect the effects of neutrino oscillations was Ray Davis' Homestake Experiment in the late 1960s, which observed a deficit in the flux of solar neutrinos ν_e
    - Verified and quantified in experiments at the Sudbury Neutrino Observatory



- A number of experimental hints and, almost literally, innumerably many theoretical speculations about other phenomena



Top Open Questions in Physics

Excerpts from the top-10, or top-24, or …

- ***What is dark matter?***

  There seems to be a halo of mysterious invisible material engulfing galaxies, which is commonly referred to as dark matter. Existence of dark (=invisible) matter is inferred from the observation of its gravitational pull, which causes the stars in the outer regions of a galaxy to orbit faster than they would if there was only visible matter present. Another indication is that we see galaxies in our own local cluster moving toward each other.

- ***What is dark energy?***

  The discovery of dark energy goes back to 1998. A group of scientists had recorded several dozen supernovae, including some so distant that their light had started to travel toward Earth when the universe was only a fraction of its present age. Contrary to their expectation, the scientists found that the expansion of the universe is not slowing, but accelerating.

  (The leaders of these teams shared the 2011 Nobel Prize in Physics.)

Excerpts from the top-10, or top-24, or …

- *What is the lifetime of the proton and how do we understand it?*

  It used to be considered that protons, unlike, say, neutrons, live forever, never decaying into smaller pieces. Then in the 1970's, theorists realized that their candidates for a grand unified theory, merging all the forces except gravity, implied that protons must be unstable. Wait long enough and, very occasionally, one should break down. Must Grand Unification work this way?

- *What physics explains the enormous disparity between the gravitational scale and the typical mass scale of the elementary particles?*

  In other words, why is gravity so much weaker than the other forces, like electromagnetism? A magnet can pick up a paper clip even though the gravity of the whole earth is pulling back on the other end.

Excerpts from the top-10, or top-24, or …

- *Can we quantitatively understand quark and gluon confinement in quantum chromodynamics and the existence of a mass gap?*

  Quantum chromodynamics, or QCD, is the theory describing the strong nuclear force. Carried by gluons, it binds quarks into particles like protons and neutrons. Apparently, the tiny subparticles are permanently confined: one can't pull a quark or a gluon from a proton because the strong force gets stronger with distance and snaps them right back inside.

Quantum Chromodynamics

What is QCD?

- Lagrangian of QCD
  - *G* = gluon fields
  - *Ψ* = quark fields

$$\mathcal{L}_{\rm QCD} = \bar{\psi}_i \left(i\gamma^\mu (D_\mu)_{ij} - m\,\delta_{ij}\right)\psi_j - \frac{1}{4}G^a_{\mu\nu}G_a^{\mu\nu}$$
$$= \bar{\psi}_i(i\gamma^\mu\partial_\mu - m)\psi_i - gG^a_\mu\bar{\psi}_i\gamma^\mu T^a_{ij}\psi_j - \frac{1}{4}G^a_{\mu\nu}G_a^{\mu\nu}$$

- The key to complexity in QCD … gluon field strength tensor

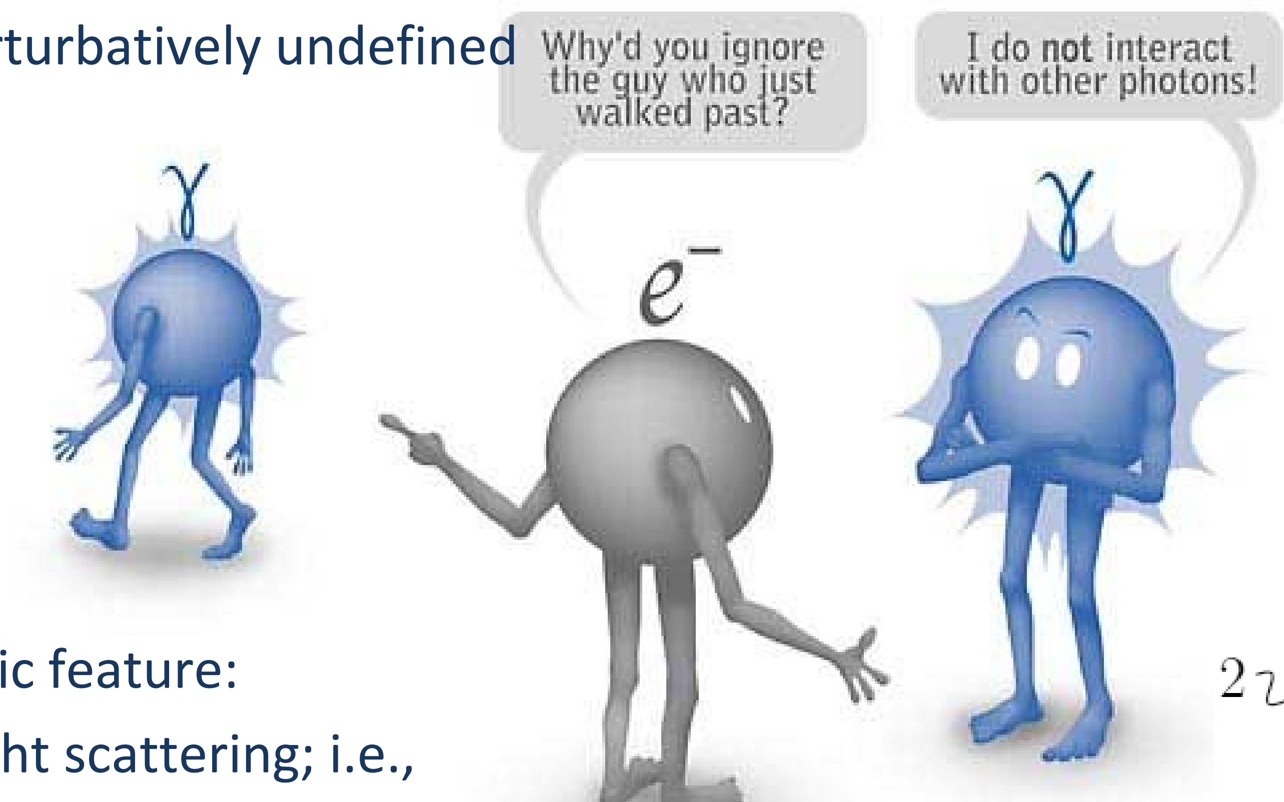
$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g f^{abc} G^a_\mu G^b_\nu$$

- Generates gluon self-interactions, whose consequences are quite extraordinary

cf.Quantum Electrodynamics

- QED is the archetypal gauge field theory
- Perturbatively simple

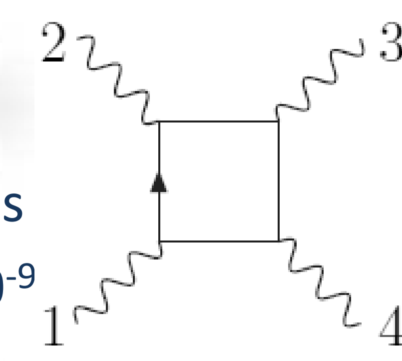
  but nonperturbatively undefined



- Chracteristic feature:

  Light-by-light scattering; i.e.,

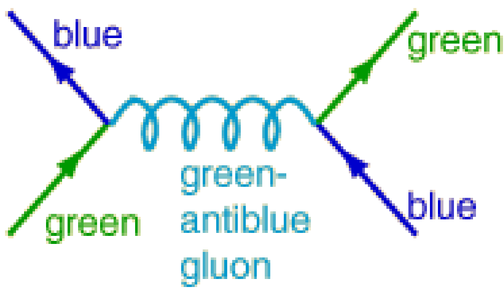
  photon-photon interaction – leading-order contribution takes place at order α^4. Extremely small probability because $\alpha^4 \approx 10^{-9}$



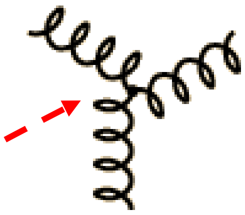
What is QCD?

Relativistic Quantum Gauge Field Theory:

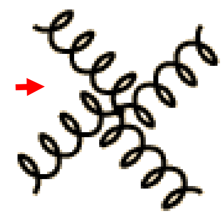
- Interactions mediated by vector boson exchange
- Vector bosons are *perturbatively*-massless



Feynman diagram for an interaction between quarks generated by a gluon.



3-gluon vertex

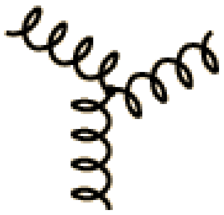


4-gluon vertex

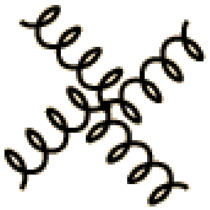
- Similar interaction in QED
- Special feature of QCD – gluon self-interactions

What is QCD?

- Novel feature of QCD
  - Tree-level interactions between gauge-bosons
  - $O(\alpha_s)$ cross-section cf. $O(\alpha_{em}{}^4)$ in QED
- One might guess that this is going to have a big impact
- Elucidating part of that impact is the origin of the 2004 Nobel Prize to Politzer, and Gross & Wilczek

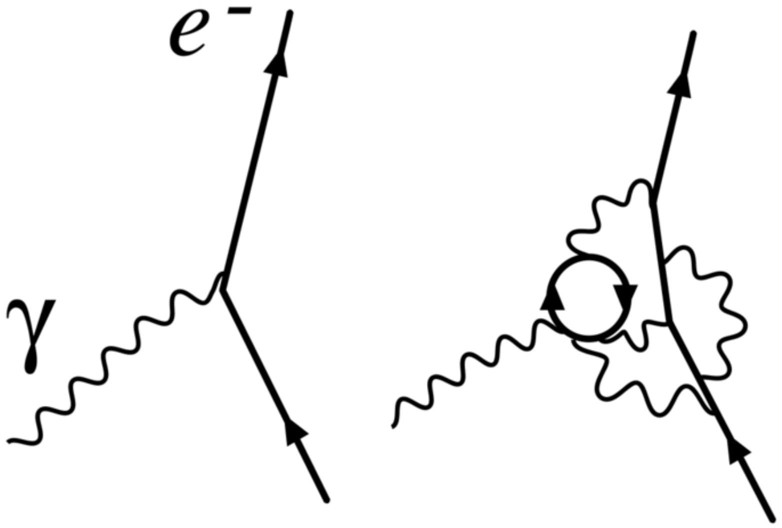


3-gluon vertex



4-gluon vertex

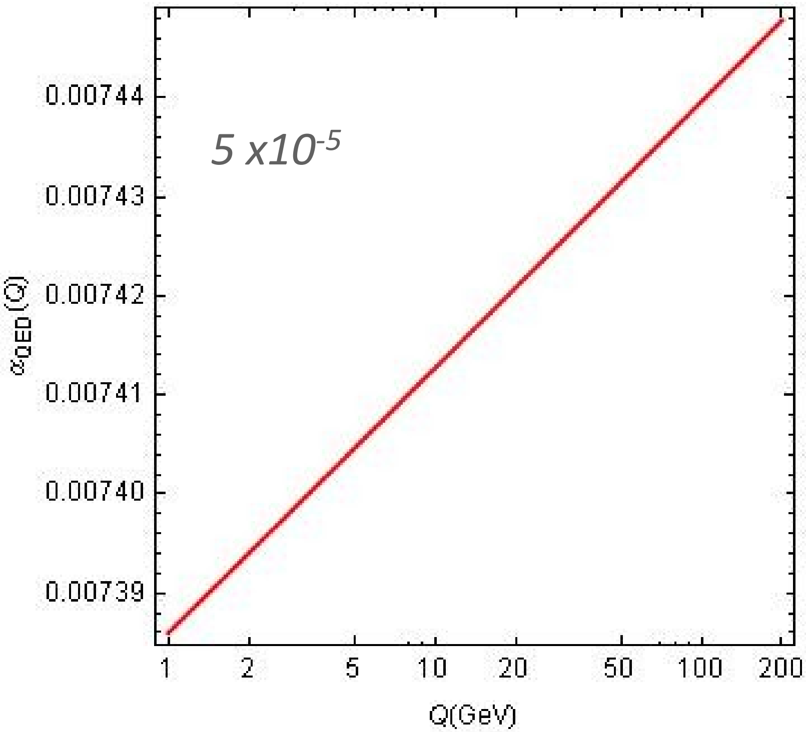
Running couplings

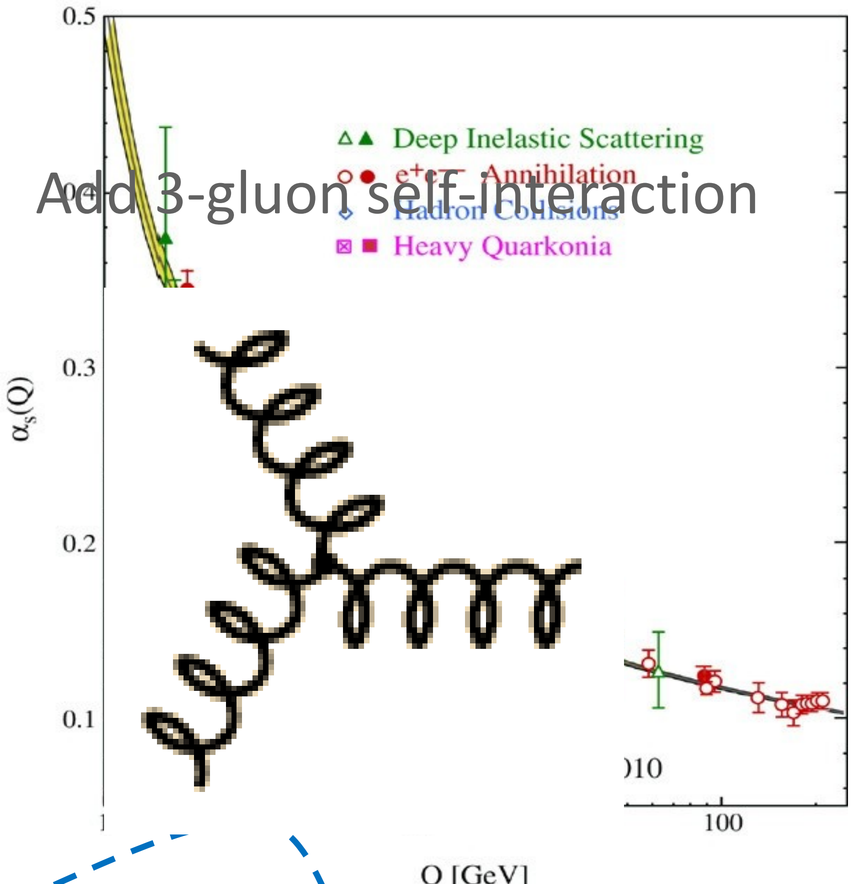


- Quantum gauge-field theories are all typified by the feature that *Nothing is Constant*
- Distribution of charge and mass, the number of particles, etc., indeed, all the things that quantum mechanics holds fixed, depend upon the wavelength of the tool used to measure them
  - particle number is not conserved in quantum field theory
- Couplings and masses are renormalised via processes involving virtual-particles. Such effects make these quantities depend on the energy scale at which one observes them

QED cf. QCD?

✓ 2004 Nobel Prize in Physics : Gross, Politzer and Wilczek





Add 3-gluon self-interaction

$$\alpha_{QED}(Q) = \frac{\alpha}{1 - \frac{2\alpha}{3\pi}\ln\frac{Q}{m_e}}$$

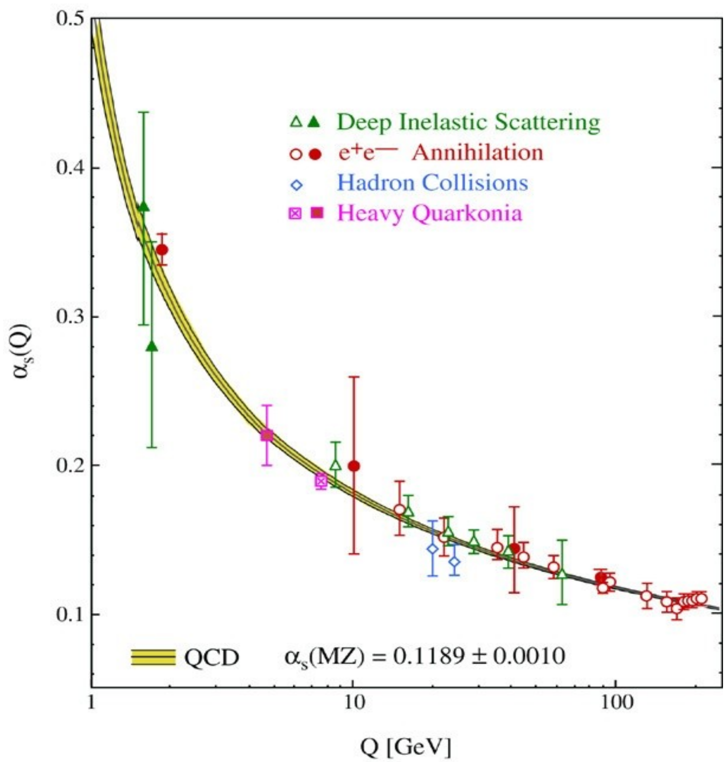
gluon antiscreening

fermion screening

$$\alpha_{QCD}(Q) = \frac{6\pi}{(33 - 2N_f)\ln\frac{Q}{\Lambda}}$$

What is QCD?

- This momentum-dependent coupling translates into a coupling that depends strongly on separation.
- Namely, the interaction between quarks, between gluons, and between quarks and gluons grows rapidly with separation
- Coupling is *huge* at separations $r = 0.2fm \approx \frac{1}{4}\, r_{proton}$



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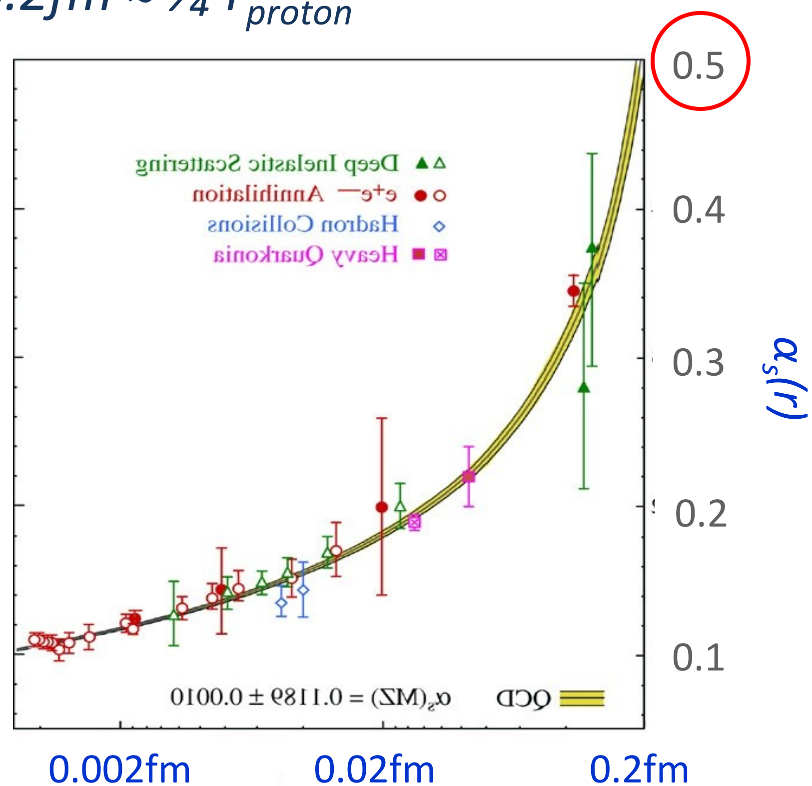
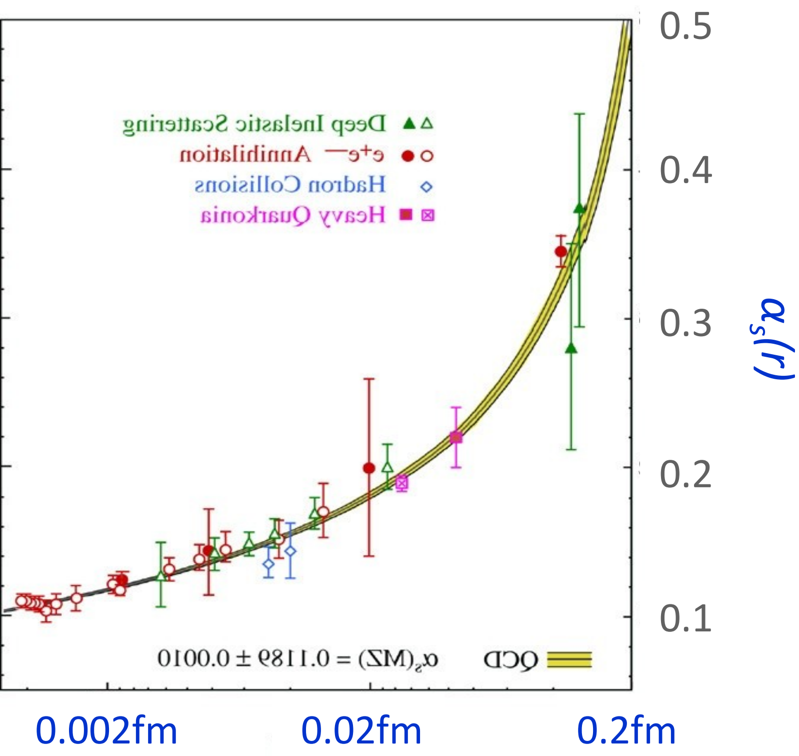




Confinement in QCD

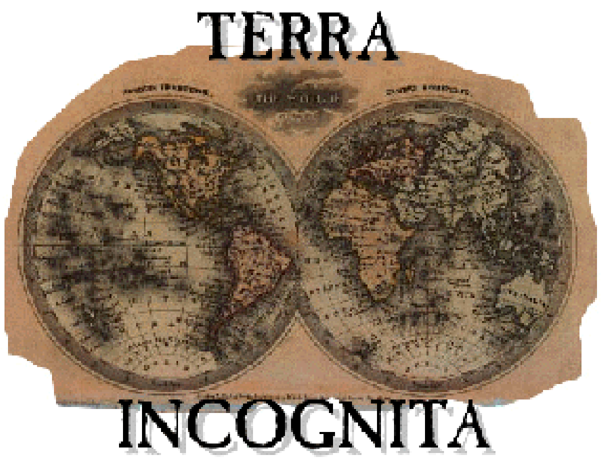
- A *peculiar* circumstance; viz., an interaction that becomes stronger as the participants try to separate
- If coupling grows so strongly with separation, then
  - perhaps it is unbounded?
  - perhaps it would require an infinite amount of energy in order to extract a quark or gluon from the interior of a hadron?
- The Confinement Hypothesis:

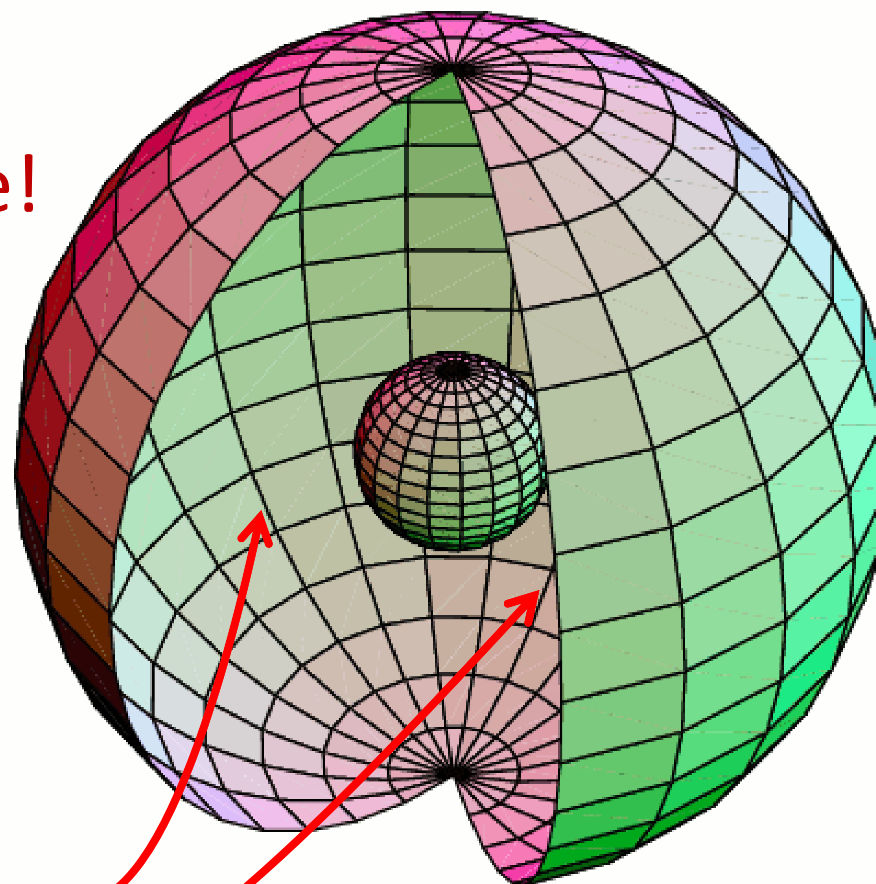
  Colour-charged particles cannot be isolated and therefore cannot be directly observed. They clump together in colour-neutral bound-states
- This is an empirical fact.

The Problem with QCD

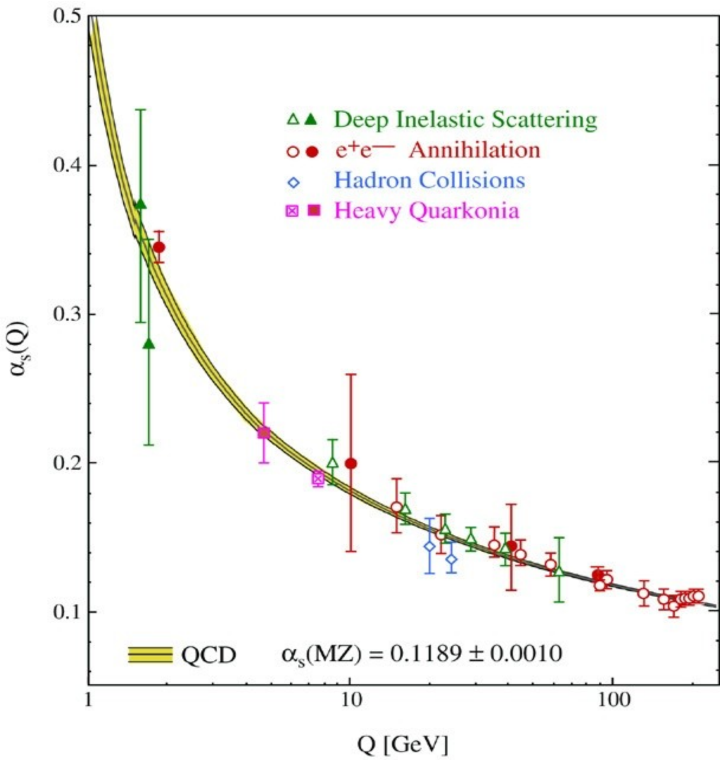
- Perhaps?!
- What we know unambiguously …

  Is that we know too little!





What is the interaction throughout more than 98% of the proton's volume?



Strong-interaction: QCD

- Asymptotically free
  - Perturbation theory is valid and accurate tool at large-Q^2
  - Hence chiral limit is defined
- *Essentially* nonperturbative for $Q^2 < 2$ GeV^2

- *Nature's only example of truly nonperturbative, fundamental theory*
- A-priori, *no idea as to what such a theory can produce*

Yang–Mills Existence and Mass Gap. *Prove that for any compact simple gauge group G, a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$. Existence includes establishing axiomatic properties at least as strong as those cited in* [**45, 35**].

Millennium prize of \$1,000,000 for proving that $SU_c(3)$ gauge theory is mathematically well-defined, which will necessarily prove or disprove the confinement conjecture

5. Comments

An important consequence of the existence of a mass gap is clustering: Let $\vec{x} \in \mathbb{R}^3$ denote a point in space. We let H and $\vec{P}$ denote the energy and momentum, generators of time and space translation. For any positive constant $C < \Delta$ and for any local quantum field operator $\mathcal{O}(\vec{x}) = e^{-i\vec{P}\cdot\vec{x}}\mathcal{O}e^{i\vec{P}\cdot\vec{x}}$ such that $\langle\Omega, \mathcal{O}\Omega\rangle = 0$, one has

$$|\langle\Omega, \mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\Omega\rangle| \leq \exp(-C|\vec{x} - \vec{y}|), \quad (2)$$

as long as $|\vec{x} - \vec{y}|$ is sufficiently large. Clustering is a locality property that, roughly speaking, may make it possible to apply mathematical results established on $\mathbb{R}^4$ to any 4-manifold, as argued at a heuristic level (for a supersymmetric extension of four-dimensional gauge theory) in [**49**]. Thus the mass gap not only has a physical significance (as explained in the introduction), but it may also be important in mathematical applications of four-dimensional quantum gauge theories to geometry. In addition the existence of a uniform gap for finite-volume approximations may play a fundamental role in the proof of existence of the infinite-volume limit.

There are many natural extensions of the Millennium problem. Among other things, one would like to prove the existence of an isolated one-particle state (an upper gap, in addition to the mass gap), to prove confinement, to

Confinement?

The study of nonperturbative QCD is the puriew of …

Hadron Physics

Hadron: Any of a class of subatomic particles that are composed of quarks and/or gluons and take part in the strong interaction.

Examples: proton, neutron, & pion.

International Scientific Vocabulary:

hadr- thick, heavy (from Greek *hadros* thick) + ^{2}on

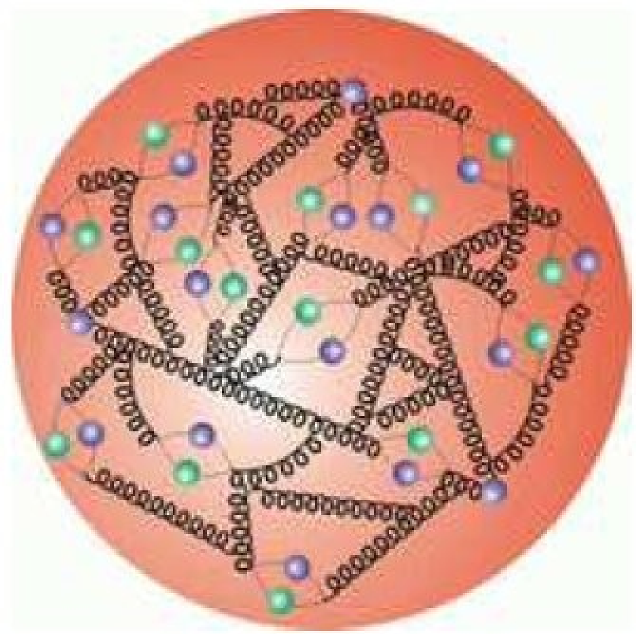
First Known Use: 1962

Baryon: hadron with half-integer-spin

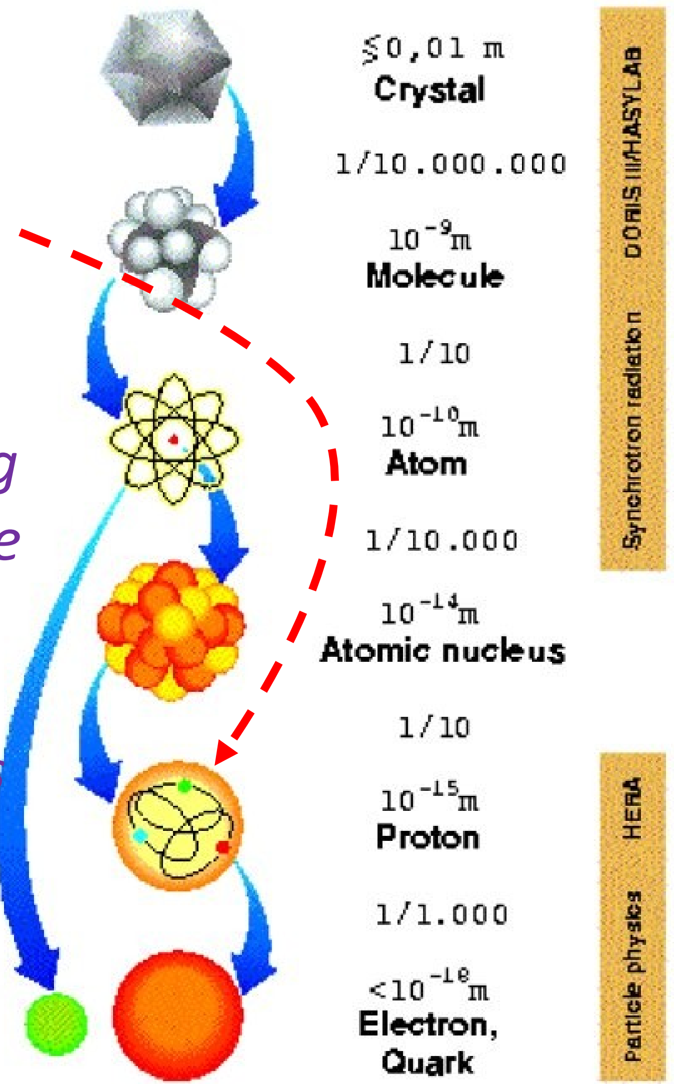
Meson: hadron with integer-spin

Hadrons

Hadron Physics



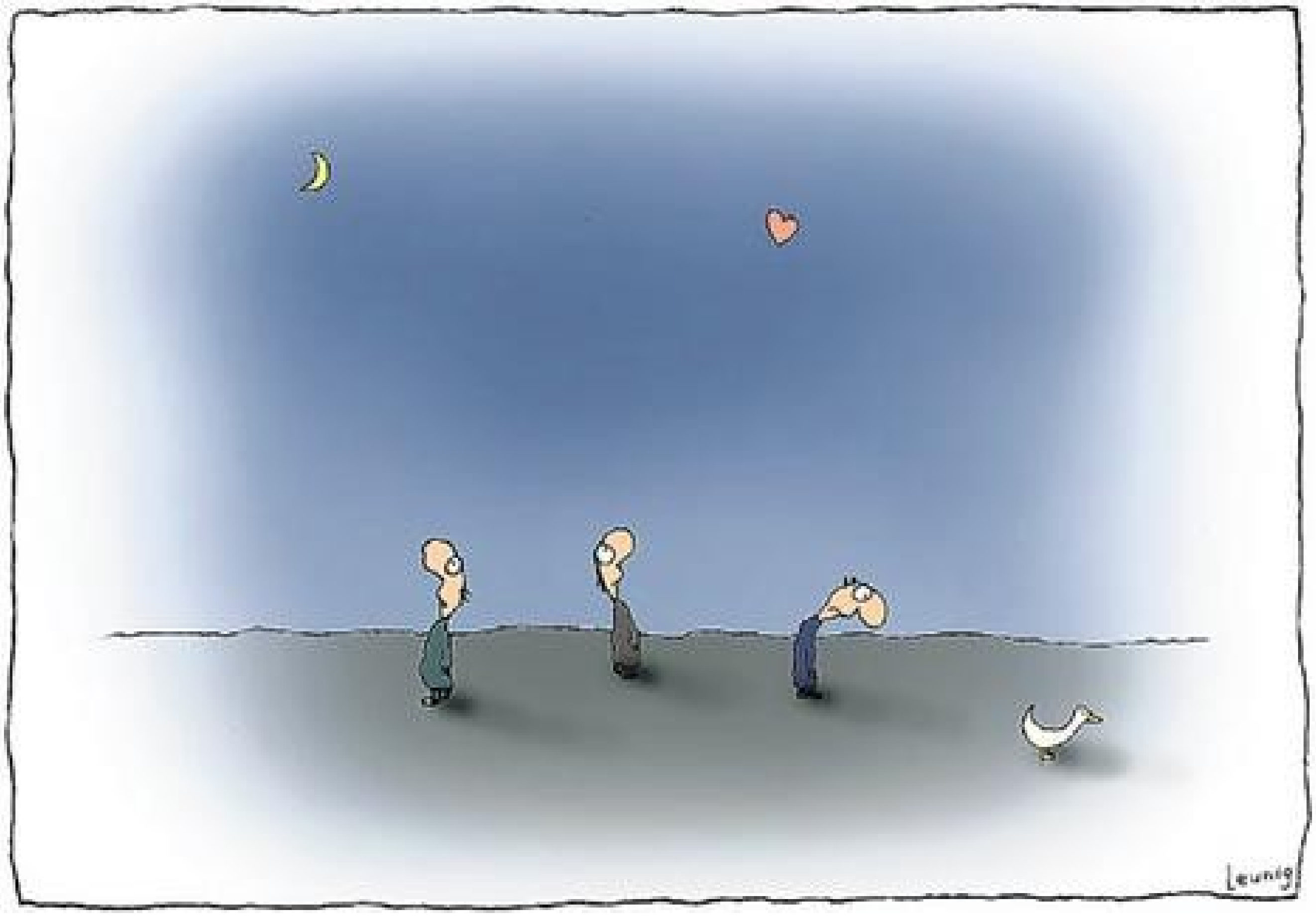
"Hadron physics is unique at the cutting edge of modern science because Nature has provided us with just one instance of a fundamental strongly-interacting theory; i.e., Quantum Chromodynamics (QCD). The community of science has never before confronted such a challenge as solving this theory."



Nuclear Science Advisory Council 2007 – Long Range Plan

"A central goal of (the DOE Office of) Nuclear Physics is to understand the structure and properties of protons and neutrons, and ultimately atomic nuclei, in terms of the quarks and gluons of QCD."

- Internationally, this is an approximately $1-billion/year effort in experiment and theory, with roughly $375-million/year in the USA.
  - Roughly 90% of these funds are spent on experiment
  - $1-billion/year is the order of the operating budget of CERN



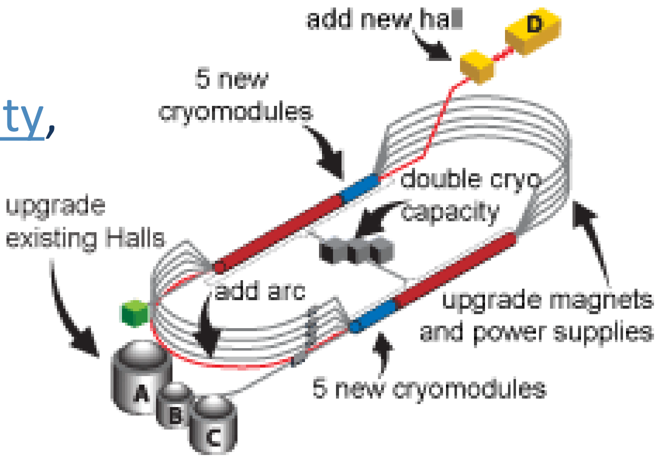
Facilities

Facilities
QCD Machines

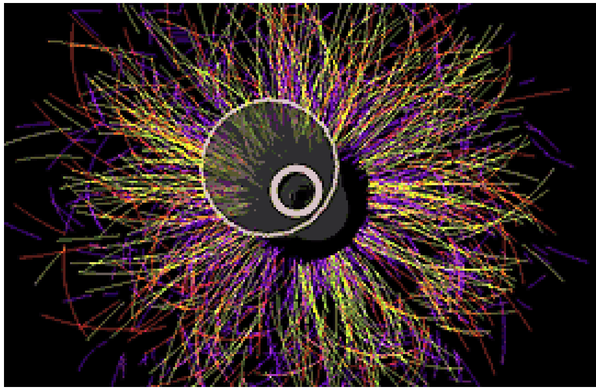
- China
  - Beijing Electron-Positron Collider
- Germany
  - COSY (Jülich Cooler Synchrotron)
  - ELSA (Bonn Electron Stretcher and Accelerator)
  - MAMI (Mainz Microtron)
  - Facility for Antiproton and Ion Research, under construction near Darmstadt. New generation experiments in 2015 (perhaps)
- Japan
  - J-PARC (Japan Proton Accelerator Research Complex), under construction in Tokai-Mura, 150km NE of Tokyo. New generation experiments to begin toward end-2012
  - KEK: Tsukuba, Belle Collaboration
- Switzerland (CERN)
  - Large Hadron Collider: ALICE Detector and COMPASS Detector "Understanding deconfinement and chiral-symmetry restoration"

Facilities
QCD Machines

- USA
  - Thomas Jefferson National Accelerator Facility,
    Newport News, Virginia
    Nature of cold hadronic matter
    Upgrade underway
      Construction cost $310-million
      New generation experiments in 2016



  - Relativistic Heavy Ion Collider, Brookhaven National Laboratory,
    Long Island, New York
    Strong phase transition, 10μs after Big Bang



A three dimensional view of the calculated particle paths resulting from collisions occurring within RHIC's STAR detector

Theory Tools

Relativistic Quantum Field Theory

- A theoretical understanding of the phenomena of Hadron Physics requires the use of the full machinery of relativistic quantum field theory.
  - Relativistic quantum field theory is the ONLY known way to reconcile quantum mechanics with special relativity.
  - Relativistic quantum field theory is based on the relativistic quantum mechanics of Dirac.
- Unification of special relativity (Poincaré covariance) and quantum mechanics took some time.
  - Questions still remain as to a practical implementation of an Hamiltonian formulation of the relativistic quantum mechanics of interacting systems.
- Poincaré group has ten generators:
  - six associated with the Lorentz transformations (rotations and boosts)
  - four associated with translations
- Quantum mechanics describes the time evolution of a system with interactions. That evolution is generated by the Hamiltonian.

Relativistic Quantum Field Theory

- Relativistic quantum mechanics predicts the existence of antiparticles; i.e., the equations of relativistic quantum mechanics admit *negative energy solutions*. However, once one allows for particles with negative energy, then particle number conservation is lost:

$$E_{system} = E_{system} + (E_{p1} + E_{anti\text{-}p1}) + \ldots \textit{ ad infinitum}$$

- This is a fundamental problem for relativistic quantum mechanics – Few particle systems can be studied in relativistic quantum mechanics but the study of (infinitely) many bodies is difficult.

  No general theory currently exists.

- This feature entails that, if a theory is formulated with an interacting Hamiltonian, then boosts will fail to commute with the Hamiltonian. Hence, the state vector calculated in one momentum frame will not be kinematically related to the state in another frame.

  That makes a new calculation necessary in every frame.

Relativistic Quantum Field Theory

- Hence the discussion of scattering, which takes a state of momentum *p* to another state with momentum *p'*, is problematic. (See, e.g., B.D. Keister and W.N. Polyzou (1991), "Relativistic Hamiltonian dynamics in nuclear and particle physics," *Adv. Nucl. Phys.* ***20,*** *225.)*
- Relativistic quantum field theory is an answer. The fundamental entities are fields, which can simultaneously represent an uncountable infinity of particles;

  e.g., neutral scalar: $\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(k)\mathrm{e}^{-ik\cdot x} + a^\dagger(k)\mathrm{e}^{ik\cdot x}\right]$

  Thus, the nonconservation of particle number is not a problem. This is crucial because key observable phenomena in hadron physics are essentially connected with the existence of *virtual particles.*
- Relativistic quantum field theory has its own problems, however. The question of whether a given relativistic quantum field theory is rigorously well defined is *unsolved.*

Relativistic Quantum Field Theory

- All relativistic quantum field theories admit analysis in perturbation theory. Perturbative renormalisation is a well-defined procedure and has long been used in Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD).
- A rigorous definition of a theory, however, means proving that the theory makes sense *nonperturbatively.* This is equivalent to proving that all the theory's renormalisation constants are nonperturbatively well-behaved.
- Hadron Physics involves QCD. While it makes excellent sense perturbatively, it is not known to be a rigorously well-defined theory. Hence it cannot truly be said to be THE theory of the strong interaction (hadron physics).
- Nevertheless, physics does not wait on mathematics. Physicists make assumptions and explore their consequences. Practitioners assume that QCD is (somehow) well-defined and follow where that leads us.

Relativistic Quantum Field Theory

- Experiment's task: explore and map the hadron physics landscape with well-understood probes, such as the electron at JLab and Mainz.
- Theory's task: employ established mathematical tools, and refine and invent others in order to use the Lagangian of QCD to predict what should be observable real-world phenomena.
- A key aim of the worlds' hadron physics programmes in experiment & theory: determine whether there are any contradictions with what we can truly *prove in* QCD.
  - Hitherto, there are none.
  - But that doesn't mean there are no puzzles nor controversies!
- Interplay between Experiment and Theory is the engine of discovery and progress. The *Discovery Potential of both is high.*
  - Much learnt in the last five years.
  - These lectures will provide a perspective on the meaning of these discoveries
- Furthermore, I expect that many surprises remain in Hadron Physics.

Dirac Equation

Dirac equation is starting point for Lagrangian formulation of quantum field theory for fermions. For a noninteracting fermion

$$[i\not\partial - m]\,\psi = 0\,, \tag{28}$$

where $\psi(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \\ u_4(x) \end{pmatrix}$ is the fermion's "spinor"

– a four component column vector, with each component spacetime dependent.

In an external electromagnetic field the fermion's wave function obeys

$$[i\not\partial - e\not A - m]\,\psi = 0\,, \tag{29}$$

obtained, as usual, via "minimal substitution:" $\boldsymbol{p}^\mu \to \boldsymbol{p}^\mu - eA^\mu$ in Eq. (28). The Dirac operator is a matrix-valued differential operator.

These equations have a manifestly Poincaré covariant appearance. A proof of covariance is given in the early chapters of: Bjorken, J.D. and Drell, S.D. (1964), *Relativistic Quantum Mechanics* (McGraw-Hill, New York).

Free particle solutions

Insert plane waves in free particle Dirac equation:

$$\psi^{(+)}(x) = \mathrm{e}^{-i(k,x)}\, u(k)\,,\ \psi^{(-)}(x) = \mathrm{e}^{+i(k,x)}\, v(k)\,,$$

and thereby obtain ...

$$(\not k - m)\, u(k) = 0\,,\ (\not k + m)\, v(k) = 0\,. \qquad (30)$$

Here there are two qualitatively different types of solution, corresponding to positive and negative energy: k & $-k$.

(Appreciation of physical reality of negative energy solutions led to prediction of antiparticles.)

Assume particle's mass is nonzero; work in rest frame:

$$(\gamma^0 - 1)\, u(m,\vec{0}) = 0\,,\ (\gamma^0 + 1)\, v(m,\vec{0}) = 0\,. \qquad (31)$$

There are clearly (remember the form of γ^0) two linearly-independent solutions of each equation:

$$u^{(1),(2)}(m,\vec{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\ v^{(1),(2)}(m,\vec{0}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Positive Energy Free Particle

Solution in arbitrary frame can be obtained via a Lorentz boost. However, simpler to observe that

$$(\not{k} - m)\,(\not{k} + m) = k^2 - m^2 = 0\,, \tag{33}$$

(The last equality is valid for real, on-shell particles.)
It follows that for arbitrary k^μ and positive energy ($E > 0$), the canonically normalised spinor is

$$u^{(\alpha)}(k) \;=\; \frac{\not{k} + m}{\sqrt{2m(m+E)}}\, u^{(\alpha)}(m,\vec{0}) = \begin{bmatrix} \left(\dfrac{E+m}{2m}\right)^{1/2} & \phi^\alpha(m,\vec{0}) \\ \dfrac{\sigma \cdot k}{\sqrt{2m(m+E)}} & \phi^\alpha(m,\vec{0}) \end{bmatrix}, \tag{34}$$

with the two-component spinors, obviously to be identified with the fermion's spin in the rest frame (the only frame in which spin has its naive meaning)

$$\phi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\; \phi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{35}$$

Negative Energy Free Particle

For negative energy: $\hat{E} = -E > 0$,

$$v^{(\alpha)}(k) = \frac{-\not{k} + m}{\sqrt{2m(m+\hat{E})}} v^{(\alpha)}(m,\vec{0}) = \begin{pmatrix} \dfrac{\sigma \cdot k}{\sqrt{2m(m+\hat{E})}} & \chi^{\alpha}(m,\vec{0}) \\ \left(\dfrac{\hat{E}+m}{2m}\right)^{1/2} & \chi^{\alpha}(m,\vec{0}) \end{pmatrix}, \quad (36)$$

with $\chi^{(\alpha)}$ obvious analogues of $\phi^{(\alpha)}$ in Eq. (35).

NB. For $\vec{k} \sim 0$ (rest frame) the lower component of the positive energy spinor is small, as is the upper component of the negative energy spinor $\Rightarrow$ Poincaré covariance, which requires the four component form, becomes important with increasing $|\vec{k}|$; indispensable for $|\vec{k}| \gtrsim m$.

NB. Solving $\vec{k} \neq 0$ equations this way works because it is clear that there are two, and only two, linearly-independent solutions of the momentum space free-fermion Dirac equations, Eqs. (30), and, for the homogeneous equations, any two covariant solutions with the correct limit in the rest-frame must give the correct boosted form.

Conjugate Spinor

In quantum field theory, as in quantum mechanics, one needs a conjugate state to define an inner product.

For fermions in Minkowski space that conjugate is $\boxed{\bar{\psi}(x) := \psi^\dagger(x)\gamma^0}$, and

$$\bar{\psi}(i \overleftarrow{\not\partial} + m) = 0\,. \tag{37}$$

This yields the following free particle spinors in momentum space (using $\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu$, a relation that is particularly important in the discussion of intrinsic parity)

$$\bar{u}^{(\alpha)}(k) = \bar{u}^{(\alpha)}(m,\vec{0})\,\frac{\not k + m}{\sqrt{2m(m+E)}} \tag{38}$$

$$\bar{v}^{(\alpha)}(k) = \bar{v}^{(\alpha)}(m,\vec{0})\,\frac{-\not k + m}{\sqrt{2m(m+E)}}\,, \tag{39}$$

Orthonormalisation

$$\begin{array}{ll} \bar{u}^{(\alpha)}(k)\,u^{(\beta)}(k) = \delta_{\alpha\beta} & \bar{u}^{(\alpha)}(k)\,v^{(\beta)}(k) = 0 \\ \bar{v}^{(\alpha)}(k)\,v^{(\beta)}(k) = -\delta_{\alpha\beta} & \bar{v}^{(\alpha)}(k)\,u^{(\beta)}(k) = 0 \end{array}\,. \tag{40}$$

Positive Energy Projection Operator

Can now construct positive energy projection operators. Consider

$$\Lambda_+(k) \quad := \quad \textstyle\sum_{\alpha=1,2} u^{(\alpha)}(k) \otimes \bar{u}^{(\alpha)}(k)\,. \tag{41}$$

Plain from the orthonormality relations, Eqs. (40), that

$$\Lambda_+(k)\, u^{(\alpha)}(k) = u^{(\alpha)}(k)\,,\ \Lambda_+(k)\, v^{(\alpha)}(k) = 0\,. \tag{42}$$

Now, since $\sum_{\alpha=1,2} u^{(\alpha)}(m,\vec{0}) \otimes \bar{u}^{(\alpha)}(m,\vec{0}) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right) = \dfrac{1+\gamma^0}{2}$, then

$$\Lambda_+(k) \quad = \quad \frac{1}{2m(m+E)}\,(\not k + m)\,\frac{1+\gamma^0}{2}\,(\not k + m)\,. \tag{43}$$

Noting that for $k^2 = m^2$; i.e., on shell,

$$(\not k + m)\,\gamma^0\,(\not k + m) = 2E\,(\not k + m),\ (\not k + m)\,(\not k + m) = 2m\,(\not k + m),$$

one finally arrives at the simple closed form:

$$\Lambda_+(k) = \frac{\not k + m}{2\,m}\,. \tag{44}$$

Negative Energy Projection Operator

The negative energy projection operator is

$$\Lambda_-(k) \quad := \quad -\sum\nolimits_{\alpha=1,2} v^{(\alpha)}(k) \otimes \bar{v}^{(\alpha)}(k) = \frac{-\not{k} + m}{2\,m}\,. \tag{45}$$

The projection operators have the following characteristic and important properties:

$$\Lambda_\pm^2(k) \quad = \quad \Lambda_\pm(k)\,, \tag{46}$$

$$\mathrm{tr}\,\Lambda_\pm(k) \quad = \quad 2\,, \tag{47}$$

$$\Lambda_+(k) + \Lambda_-(k) \quad = \quad 1\,. \tag{48}$$

Green Functions / Propagators

The Dirac equation is a partial differential equation.
A general method for solving such equations is to use a Green function, which is the inverse of the differential operator that appears in the equation.
The analogy with matrix equations is obvious and can be exploited heuristically.

Dirac equation, Eq. (29): $[i\not\partial_x - e\not A(x) - m]\,\psi(x) = 0\,,$ yields the wave function for a fermion in an external electromagnetic field.

Consider the operator obtained as a solution of the following equation

$$[i\not\partial_{x'} - e\not A(x') - m]\,S(x',x) = \mathbf{1}\,\delta^4(x'-x)\,. \qquad (49)$$

Obviously if, at a given spacetime point x, $\psi(x)$ is a solution of Eq. (29), then

$$\psi(x') := \int d^4x\, S(x',x)\,\psi(x) \qquad (50)$$

is a solution of... $[i\not\partial_{x'} - e\not A(x') - m]\,\psi(x') = 0\,;$ (51)

i.e., $S(x',x)$ has *propagated* the solution at x to the point x'.

Analogue of Huygens Principle in Wave Mechanics

Green Functions / Propagators

This approach is practical because all physically reasonable external fields can only be nonzero on a compact subdomain of spacetime.

Therefore the solution of the complete equation is transformed into solving for the Green function, which can then be used to propagate the free-particle solution, already found, to arbitrary spacetime points.

However, obtaining the *exact* form of $S(x', x)$ is *impossible* for all but the simplest cases (see, e.g.,

- Dittrich, W. and Reuter, M. (1985), *Effective Lagrangians in Quantum Electrodynamics* (Springer Verlag, Berlin);
- Dittrich, W. and Reuter, M. (1985), *Selected Topics in Gauge Theories* (Springer Verlag, Berlin).)

This is where and why perturbation theory so often rears its not altogether handsome head.

Free Fermion Propagator

In the absence of an external field the Green Function equation, Eq. (49), becomes

$$[i\partial\!\!\!/_{x'} - m]\, S(x',x) = 1\, \delta^4(x'-x)\,. \tag{55}$$

Assume a solution of the form:

$$S_0(x',x) = S_0(x'-x) = \int \frac{d^4p}{(2\pi)^4}\, \mathrm{e}^{-i(p,x'-x)}\, S_0(p)\,, \tag{56}$$

so that substituting yields

$$(p\!\!\!/ - m)\, S_0(p) = 1\,;\ \text{i.e.,}\ S_0(p) = \frac{p\!\!\!/ + m}{p^2 - m^2}\,. \tag{57}$$

To obtain the result in configuration space one must adopt a prescription for handling the on-shell singularities in $S(p)$ at $p^2 = m^2$.

- That convention is tied to the boundary conditions applied to Eq. (55).
- An obvious and physically sensible definition of the Green function is that it should propagate
  - positive-energy-fermions and -antifermions forward in time but not backward in time,
  - and vice versa for negative energy states.

Feynman's Fermion Propagator

The wave function for a positive energy free-fermion is

$$\psi^{(+)}(x) = u(p)\, \mathrm{e}^{-i(p,x)} \,. \tag{58}$$

The wave function for a positive-energy antifermion is the charge-conjugate of the negative-energy fermion solution ($C = i\gamma^2\gamma^0$ and $(\cdot)^{\mathrm{T}}$ denotes matrix transpose):

$$\psi_c^{(+)}(x) = C\,\gamma^0\,[v(p)\,\mathrm{e}^{i(p,x)}]^* = C\,\bar{v}(p)^{\mathrm{T}}\,\mathrm{e}^{-i(p,x)} \,, \tag{59}$$

Follows from properties of spinors and projection operators that our physically sensible $S_0(x'-x)$ must contain only positive-frequency components for $t = x_0' - x_0 > 0$; i.e., in this case it must be proportional to $\Lambda_+(p)$.

- Exercise: Verify this.

Can ensure this via a small modification of the denominator of Eq. (57), with $\eta \to 0^+$ at the end of all calculations:

$$S_0(p) = \frac{\not p + m}{p^2 - m^2} \to \frac{\not p + m}{p^2 - m^2 + i\eta} \,. \tag{60}$$

(This prescription defines the **Feynman** propagator.)

Green Function - Interacting Theory

Eq. (49), Green function for a fermion in an external electromagnetic field:

$$[i\not\partial_{x'} - e\not A(x') - m]\, S(x',x) = 1\,\delta^4(x'-x)\,, \tag{61}$$

A closed form solution of this equation is impossible in all but the simplest field configurations. Is there, nevertheless, a way to construct an approximate solution that can systematically be improved?

One Answer: Perturbation Theory – rewrite the equation:

$$[i\not\partial_{x'} - m]\, S(x',x) = 1\,\delta^4(x'-x) + e\not A(x')\, S(x',x)\,, \tag{62}$$

which, as can easily be seen by substitution (Verify This), is solved by

$$\begin{aligned} S(x',x) &= S_0(x'-x) + e\int d^4y\, S_0(x'-y)\not A(y)\, S(y,x) \\ &= S_0(x'-x) + e\int d^4y\, S_0(x'-y)\not A(y)\, S_0(y-x) \\ &\quad + e^2\int d^4y_1 \int d^4y_2\, S_0(x'-y_1)\not A(y_1)\, S_0(y_1-y_2)\not A(y_2)\, S_0(y_2-x) \\ &\quad + \ldots \end{aligned} \tag{63}$$

Green Function - Interacting Theory

- This perturbative expansion of the full propagator in terms of the free propagator provides an archetype for perturbation theory in quantum field theory.
  - One obvious application is the scattering of an electron/positron by a Coulomb field, which is an example explored in Sec. 2.5.3 of Itzykson, C. and Zuber, J.-B. (1980), *Quantum Field Theory (McGraw-Hill, New York).*
  - Equation (63) is a first example of a Dyson-Schwinger equation.
- This Green function has the following interpretation
  1. It creates a positive energy fermion (antifermion) at spacetime point x;
  2. Propagates the fermion to spacetime point x'; i.e., forward in time;
  3. Annihilates this fermion at x'.
- The process can equally well be viewed as
  1. Creation of a negative energy antifermion (fermion) at spacetime point x';
  2. Propagation of the antifermion to the point x; i.e., backward in time;
  3. Annihilation of this antifermion at x.
- Other propagators have similar interpretations.

Anything troubling you?

Functional Integrals

- Local gauge theories are the keystone of contemporary hadron and high-energy physics. QCD is a local gauge theory.
- Such theories are difficult to quantise because the gauge dependence is an extra non-dynamical degree of freedom that must be dealt with.
- The modern approach is to quantise the theories using the method of functional integrals. Good references:
  - Itzykson, C. and Zuber, J.-B. (1980), *Quantum Field Theory (McGraw-Hill,*New York);
  - Pascual, P. and Tarrach, R. (1984), Lecture Notes in Physics, Vol. **194,** *QCD: Renormalization for the Practitioner (Springer-Verlag, Berlin).*
- Functional Integration replaces canonical second-quantisation.
- NB. In general mathematicians do not regard local gauge theory functional integrals as well-defined.

Dyson-Schwinger Equations

- It has long been known that from the field equations of quantum field theory one can derive a system of coupled integral equations interrelating all of a theory's Green functions:
  - Dyson, F.J. (1949), "The S Matrix In Quantum Electrodynamics," *Phys. Rev.* **75,** 1736.
  - Schwinger, J.S. (1951), "On The Green's Functions Of Quantized Fields: 1 and 2," *Proc. Nat. Acad. Sci.* ***37*** *(1951) 452; ibid 455.*
- This collection of a countable infinity of equations is called the complex of Dyson-Schwinger equations (DSEs).
- It is an intrinsically nonperturbative complex, which is vitally important in proving the renormalisability of quantum field theories. At its simplest level the complex provides a generating tool for perturbation theory.
- In the context of quantum electrodynamics (QED), I will illustrate a nonperturbative derivation of one equation in this complex. The derivation of others follows the same pattern.

Photon Vacuum Polarisation

NB. This is one part of the Lamb Shift

- Action for QED with N_f flavours of electromagnetically active fermions:

$$S[A_\mu, \psi, \bar\psi] = \int d^4x \left[\sum_{f=1}^{N_f} \bar\psi^f \left(i\partial\!\!\!/ - m_0^f + e_0^f A\!\!\!/ \right) \psi^f - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\lambda_0} \partial^\mu A_\mu(x)\, \partial^\nu A_\nu(x) \right] \quad (64)$$

- Manifestly Poincaré invariant action:
  - $\bar\psi^f(x)$, $\psi^f(x)$ are elements of Grassmann algebra that describe the fermion degrees of freedom;
  - m_0^f are the fermions' bare masses and e_0^f, their charges;
  - and $A_\mu(x)$ describes the gauge boson [photon] field, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and λ_0 the bare gauge fixing parameter. (NB. To describe an electron the physical charge $e_f < 0$.)

QED Generating Functional

The Generating Functional is defined via the QED action

$$W[J_\mu, \xi, \bar{\xi}] = \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar{\psi}]$$

$$\times \exp\left\{ i \int d^4x \left[-\frac{1}{4} F^{\mu\nu}(x)\, F_{\mu\nu}(x) - \frac{1}{2\lambda_0}\, \partial^\mu A_\mu(x)\, \partial^\nu A_\nu(x) \right.\right.$$

$$+ \sum_{f=1}^{N_f} \bar{\psi}^f \left(i\partial\!\!\!/ - m_0^f + e_0^f A\!\!\!/ \right) \psi^f$$

$$\left.\left. + J^\mu(x) A_\mu(x) + \bar{\xi}^f(x)\psi^f(x) + \bar{\psi}^f(x)\xi^f(x) \right] \right\}, \qquad (65)$$

- simple interaction term: $\boxed{\bar{\psi}^f e_0^f A\!\!\!/ \psi^f}$
- J_μ is an external source for the electromagnetic field
- ξ^f, $\bar{\xi}^f$ are external sources for the fermion field that, of course, are elements in the Grassmann algebra.

The Generating Functional expresses ***every*** feature of the theory.

Functional Field Equations

Advantageous to work with the generating functional of connected Green functions; i.e., $Z[J_\mu, \bar{\xi}, \xi]$ defined via

$$W[J_\mu, \xi, \bar{\xi}] =: \exp\left\{ iZ[J_\mu, \xi, \bar{\xi}] \right\}. \tag{64}$$

Derivation of a DSE follows simply from observation that the integral of a total derivative vanishes, given appropriate boundary conditions; e.g.,

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} dx \boxed{\frac{d}{dx} f(x)} \\ &= f(\infty) - f(-\infty) \end{aligned}$$

so long as $f(\infty) = f(-\infty)$, which includes $f(\infty) = f(-\infty) = 0$; viz., the case of "fields" that vanish far from the interaction domain, centred on $x = 0$.

Last line has meaning as a functional differential operator acting on the generating functional.

Functional Field Equations

Advantageous to work with the generating functional of connected Green functions; i.e., $Z[J_\mu, \bar\xi, \xi]$ defined via

$$W[J_\mu,\xi,\bar\xi] =: \exp\left\{ iZ[J_\mu,\xi,\bar\xi]\right\}. \tag{64}$$

Derivation of a DSE follows simply from observation that the integral of a total derivative vanishes, given appropriate boundary conditions; e.g.,

$$\begin{aligned} 0 &= \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar\psi]\,\frac{\delta}{\delta A_\mu(x)}\,\mathrm{e}^{i\left(S[A_\mu,\psi,\bar\psi]+\int d^4x\,[\bar\psi^f\xi^f+\bar\xi^f\psi^f+A_\mu J^\mu]\right)} \\ &= \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar\psi]\left\{\frac{\delta S}{\delta A_\mu(x)}+J_\mu(x)\right\} \\ &\quad \times \exp\left\{ i\left(S[A_\mu,\psi,\bar\psi]+\int d^4x\left[\bar\psi^f\xi^f+\bar\xi^f\psi^f+A_\mu J^\mu\right]\right)\right\} \\ &= \left\{\frac{\delta S}{\delta A_\mu(x)}\left[\frac{\delta}{i\delta J},\frac{\delta}{i\delta\bar\xi},-\frac{\delta}{i\delta\xi}\right]+J_\mu(x)\right\} W[J_\mu,\xi,\bar\xi]\,, \end{aligned} \tag{65}$$

Craig Roberts: Emergence of DSEs in Real-World QCD IA (89)

Functional Field Equations

Differentiate Eq. (64) to obtain

$$\frac{\delta S}{\delta A_\mu(x)} = \left[\partial_\rho \partial^\rho g_{\mu\nu} - \left(1 - \frac{1}{\lambda_0}\right)\partial_\mu \partial_\nu\right] A^\nu(x) + \sum_f e_0^f\, \overline{\psi}^f(x)\gamma_\mu \psi^f(x)\,, \quad (66)$$

Equation (65) then becomes

$$-J_\mu(x) = \left[\partial_\rho \partial^\rho g_{\mu\nu} - \left(1 - \frac{1}{\lambda_0}\right)\partial_\mu \partial_\nu\right] \frac{\delta Z}{\delta J_\nu(x)}$$
$$+ \sum_f e_0^f \Bigg(- \frac{\delta Z}{\delta \xi^f(x)} \gamma_\mu \frac{\delta Z}{\delta \bar{\xi}^f(x)} + \frac{\delta}{\delta \xi^f(x)} \Bigg] \gamma_\mu \frac{\delta\, iZ}{\delta \bar{\xi}^f(x)} \Bigg] \Bigg), \quad (67)$$

where we have divided through by $W[J_\mu, \xi, \bar{\xi}]$.

Equation (67) represents a compact form of the nonperturbative equivalent of Maxwell's equations

One-Particle Irreducible Green Function

Introduce generating functional for one-particle-irreducible (1PI) Green functions:

$$\Gamma[A_\mu, \psi, \bar{\psi}] \qquad (68)$$

Obtained from $Z[J_\mu, \xi, \bar{\xi}]$ via a Legendre transformation; namely,

$$Z[J_\mu, \xi, \bar{\xi}] = \Gamma[A_\mu, \psi, \bar{\psi}] + \int d^4x \left[\bar{\psi}^f \xi^f + \bar{\xi}^f \psi^f + A_\mu J^\mu \right]. \qquad (69)$$

NB. On the right-hand-side, $\boldsymbol{A_\mu}$, $\boldsymbol{\psi}$, $\boldsymbol{\bar{\psi}}$ are functionals of the sources.

One-particle-irreducible n-point function or "proper vertex" contains no contributions that become disconnected when a single connected m-point Green function is removed; e.g., via functional differentiation.

- No diagram representing or contributing to a given proper vertex separates into two disconnected diagrams if only one connected propagator is cut. (Detailed explanation: Itzykson, C. and Zuber, J.-B. (1980), *Quantum Field Theory* (McGraw-Hill, New York), pp. 289-294.)

Implications of Legendre Transformation

It is plain from the definition of the Generating Functional, Eq. (65), that

$$\frac{\delta Z}{\delta J^\mu(x)} = A_\mu(x)\,,\ \frac{\delta Z}{\delta \bar\xi(x)} = \psi(x)\,,\ \frac{\delta Z}{\delta \xi(x)} = -\bar\psi(x)\,, \tag{70}$$

where here the external sources are **nonzero**.

Hence Γ in Eq. (69) must satisfy

$$\frac{\delta \Gamma}{\delta A^\mu(x)} = -J_\mu(x)\,,\ \frac{\delta \Gamma}{\delta \bar\psi^f(x)} = -\xi^f(x)\,,\ \frac{\delta \Gamma}{\delta \psi^f(x)} = \bar\xi^f(x)\,. \tag{71}$$

(NB. Since the sources are not zero then, e.g.,

$$A_\rho(x) = A_\rho(x;[J_\mu,\xi,\bar\xi]) \ \Rightarrow\ \frac{\delta A_\rho(x)}{\delta J^\mu(y)} \neq 0\,, \tag{72}$$

with analogous statements for the Grassmannian functional derivatives.)

NB. It is easy to see that setting $\bar\psi = 0 = \psi$ after differentiating Γ gives zero *unless* there are equal numbers of $\bar\psi$ and ψ derivatives. (Integrand is odd under $\psi \to -\psi$.)

Origin of Furry's Theorem

Green Function's Inverse

- Consider the operator and matrix product (with spinor labels r, s, t)

$$-\int d^4z\, \frac{\delta^2 Z}{\delta\xi_r^f(x)\bar{\xi}_t^h(z)}\, \frac{\delta^2\Gamma}{\delta\psi_t^h(z)\overline{\psi}_s^g(y)}\Bigg|_{\substack{\xi=\bar{\xi}=0\\ \psi=\overline{\psi}=0}} . \qquad (73)$$

- Using Eqs. (70), (71), this simplifies as follows:

$$=\int d^4z\, \frac{\delta\psi_t^h(z)}{\delta\xi_r^f(x)}\, \frac{\delta\xi_s^g(y)}{\delta\psi_t^h(z)}\Bigg|_{\substack{\xi=\bar{\xi}=0\\ \psi=\overline{\psi}=0}} = \frac{\delta\xi_s^g(y)}{\delta\xi_r^f(x)}\Bigg|_{\psi=\overline{\psi}=0} = \delta_{rs}\,\delta^{fg}\,\delta^4(x-y) . \qquad (74)$$

- Back in Eq. (67), setting $\bar{\xi}=0=\xi$ one obtains

$$\frac{\delta\Gamma}{\delta A^\mu(x)}\Bigg|_{\psi=\overline{\psi}=0} = \left[\partial_\rho\partial^\rho g_{\mu\nu} - \left(1-\frac{1}{\lambda_0}\right)\partial_\mu\partial_\nu\right]A^\nu(x) - i\sum_f e_0^f \mathrm{tr}\left[\gamma_\mu S^f(x,x;[A_\mu])\right] , \qquad (75)$$

Identification: $$S^f(x,y;[A_\mu]) = -\frac{\delta^2 Z}{\delta\xi^f(y)\bar{\xi}^f(x)} = \frac{\delta^2 Z}{\delta\bar{\xi}^f(x)\xi^f(y)} \text{ (no summation on } f\text{)} , \qquad (76)$$

Green Function's Inverse

As a direct consequence of Eq. (73) the inverse of this Green function is given by

$$S^f(x,y;[A])^{-1} = \left.\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar{\psi}^f(y)}\right|_{\psi=\bar{\psi}=0} . \qquad (77)$$

General property: functional derivatives of the generating functional for 1PI Green functions are related to the associated propagator's inverse.

Clearly, the in-vacuum fermion propagator or, another name, the connected fermion 2-point function is

$$S^f(x,y) := S^f(x,y;[A_\mu=0]) . \qquad (78)$$

Such vacuum Green functions are keystones in quantum field theory.

To continue, differentiate Eq. (75) with respect to $A_\nu(y)$ and set $J_\mu(x)=0$:

$$\left.\frac{\delta^2\Gamma}{\delta A^\mu(x)\delta A^\nu(y)}\right|_{\substack{A_\mu=0\\ \psi=\bar{\psi}=0}} = \left[\partial_\rho\partial^\rho g_{\mu\nu} - \left(1-\frac{1}{\lambda_0}\right)\partial_\mu\partial_\nu\right]\delta^4(x-y)$$
$$-i\sum_f e_0^f \operatorname{tr}\left[\gamma_\mu \frac{\delta}{\delta A_\nu(y)}\left(\left.\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar{\psi}^f(x)}\right|_{\psi=\bar{\psi}=0}\right)^{-1}\right] . \qquad (79)$$

Inverse of Photon Propagator

- l.h.s. is easily understood – Eqs. (77), (78) define the inverse of the fermion propagator. Hence, l.h.s. must be the inverse of the photon propagator:

$$(D^{-1})^{\mu\nu}(x,y) := \left.\frac{\delta^2\Gamma}{\delta A^\mu(x)\delta A^\nu(y)}\right|_{\substack{A_\mu = 0 \\ \psi=\bar\psi=0}} . \tag{80}$$

- r.h.s., however, must be simplified and interpreted. First observe that

$$-\frac{\delta}{\delta A_\nu(y)}\left(\left.\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar\psi^f(x)}\right|_{\psi=\bar\psi=0}\right)^{-1} = \int d^4u d^4w \ldots \tag{81}$$

$$\left(\left.\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar\psi^f(w)}\right|_{\psi=\bar\psi=0}\right)^{-1}\frac{\delta}{\delta A_\nu(y)}\frac{\delta^2\Gamma}{\delta\psi^f(u)\delta\bar\psi^f(w)}\left(\left.\frac{\delta^2\Gamma}{\delta\psi^f(w)\delta\bar\psi^f(x)}\right|_{\psi=\bar\psi=0}\right)^{-1},$$

  - Analogue of result for finite dimensional matrices:

$$\frac{d}{dx}[A(x)A^{-1}(x) = \boldsymbol{I}] = 0 = \frac{dA(x)}{dx}A^{-1}(x) + A(x)\frac{dA^{-1}(x)}{dx}$$

$$\Rightarrow \quad \frac{dA^{-1}(x)}{dx} = -A^{-1}(x)\frac{dA(x)}{dx}A^{-1}(x) . \tag{82}$$

Proper Fermion-Photon Vertex

Equation (81) involves the 1PI 3-point function (no summation on f)

$$e_0^f \Gamma_\mu^f(x,y;z) := \frac{\delta}{\delta A_\nu(z)} \frac{\delta^2 \Gamma}{\delta \psi^f(x) \delta \bar{\psi}^f(y)} \,. \tag{83}$$

This is the proper fermion-gauge-boson vertex.

At leading order in perturbation theory

$$\Gamma_\nu^f(x,y;z) = \gamma_\nu \, \delta^4(x-z) \, \delta^4(y-z) \,, \tag{84}$$

Result can be obtained via explicit calculation of functional derivatives in Eq. (83).

Photon Vacuum Polarisation

Define the gauge-boson *vacuum polarisation*:

$$\Pi_{\mu\nu}(x,y) = i\sum_f (e_0^f)^2 \int d^4z_1\, d^4z_2\, \mathrm{tr}[\gamma_\mu S^f(x,z_1)\Gamma_\nu^f(z_1,z_2;y)S^f(z_2,x)], \quad (85)$$

Gauge-boson vacuum polarisation, or "photon self-energy,"

- Describes modification of gauge-boson's propagation characteristics owing to the presence of virtual particle-antiparticle pairs in quantum field theory.
- In particular, the photon vacuum polarisation is an important element in the description of a process such as $\rho^0 \to e^+e^-$.

Eq. (79) may now be expressed as

$$(D^{-1})^{\mu\nu}(x,y) = [\partial_\rho\partial^\rho g_{\mu\nu} - (1 - \frac{1}{\lambda_0})\partial_\mu\partial_\nu]\delta^4(x-y) + \Pi_{\mu\nu}(x,y)\,. \quad (86)$$

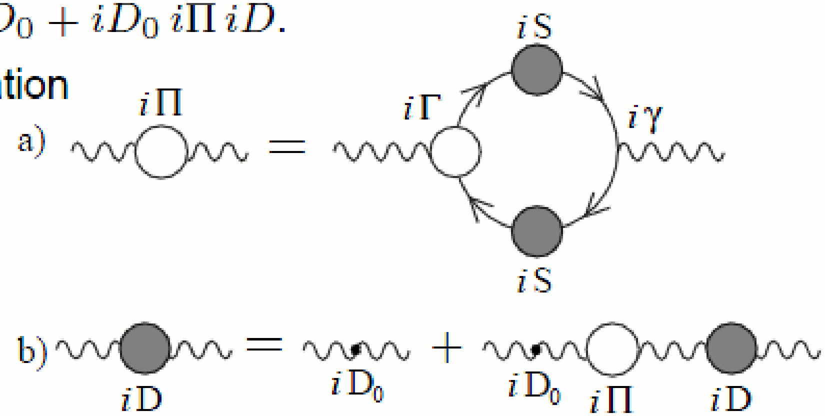
The propagator for a free gauge boson is [use $\Pi_{\mu\nu}(x,y) \equiv 0$ in Eq. (86)]

$$D_0^{\mu\nu}(q) = \frac{-g^{\mu\nu} + (q^\mu q^\nu/[q^2 + i\eta])}{q^2 + i\eta} - \lambda_0 \frac{q^\mu q^\nu}{(q^2 + i\eta)^2}\,, \quad (87)$$

DSE for Photon Propagator

Then Eq. (86) can be written $iD = iD_0 + iD_0\, i\Pi\, iD$.

- This is a Dyson-Schwinger Equation



In presence of interactions;
i.e., for $\Pi_{\mu\nu} \neq 0$ in Eq. (86),

$$D^{\mu\nu}(q) = \frac{-g^{\mu\nu} + (q^\mu q^\nu/[q^2 + i\eta])}{q^2 + i\eta}\,\frac{1}{1+\Pi(q^2)} - \lambda_0 \frac{q^\mu q^\nu}{(q^2+i\eta)^2}\,, \qquad (88)$$

Used the "Ward-Takahashi identity:" $q_\mu\, \Pi_{\mu\nu}(q) = 0 = \Pi_{\mu\nu}(q)\, q_\nu$,

$$\Rightarrow \Pi^{\mu\nu}(q) = (-g^{\mu\nu} q^2 + q^\mu q^\nu)\, \Pi(q^2)\,. \qquad (89)$$

$\Pi(q^2)$ is the polarisation scalar. Independent of the gauge parameter, λ_0, in QED.

- $\lambda_0 = 1$ is called "Feynman gauge." Useful in perturbative calculations because it simplifies the $\Pi(q^2) = 0$ gauge boson propagator enormously.
- In nonperturbative applications, however, $\lambda_0 = 0$, "Landau gauge," is most useful because it ensures that the gauge boson propagator is itself transverse.

Ward-Takahashi Identities

- Ward-Takahashi identities (WTIs) are relations satisfied by n-point Green functions, relations which are an essential consequence of a theory's local gauge invariance; i.e., local current conservation.
- They can be proved directly from the generating functional and have physical implications. For example, Eq. (89) ensures that the photon remains massless in the presence of charged fermions.
- A discussion of WTIs can be found in
  - Bjorken, J.D. and Drell, S.D. (1965), *Relativistic Quantum Fields (McGraw-Hill,* New York), pp. 299-303,
  - Itzykson, C. and Zuber, J.-B. (1980), *Quantum Field Theory (McGraw-Hill,* New York), pp. 407-411.
- Their generalisation to non-Abelian theories as "Slavnov-Taylor" identities is described in
  - Pascual, P. and Tarrach, R. (1984), Lecture Notes in Physics, Vol. **194,** *QCD: Renormalization for the Practitioner (Springer-Verlag, Berlin), Chap. 2.*

Vacuum Polarisation in Momentum Space

- In absence of external sources, Eq. (85) can easily be represented in momentum space, because then the 2- and 3-point functions appearing therein must be translationally invariant and hence they can be expressed simply in terms of Fourier amplitudes; i.e., we have

$$i\Pi_{\mu\nu}(q) = -\sum_f (e_0^f)^2 \int \frac{d^4\ell}{(2\pi)^4} \mathrm{tr}[(i\gamma_\mu)(iS^f(\ell))(i\Gamma^f(\ell,\ell+q))(iS(\ell+q))] \,. \quad (90)$$

- The reduction to a single integral makes momentum space representations most widely used in continuum calculations.
- QED: the vacuum polarisation is directly related to the running coupling constant, which is a connection that makes its importance obvious.
- QCD: connection not so direct but, nevertheless, the polarisation scalar is a key component in the evaluation of the strong running coupling.
- Observed: second derivatives of the generating functional, $\Gamma[A_\mu,\psi,\bar\psi]$, give the inverse-fermion and -photon propagators; third derivative gave the proper photon-fermion vertex. In general, all derivatives of $\Gamma[A_\mu,\psi,\bar\psi]$, higher than two, produce a proper vertex, number and type of derivatives give the number and type of proper Green functions that it can connect.

Any Questions?

Functional Dirac Equation

Equation (67) is a nonperturbative generalisation of Maxwell's equation in quantum field theory. Its derivation provides the archetype by which one can obtain an equivalent generalisation of Dirac's equation:

$$
\begin{aligned}
0 &= \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar{\psi}]\,\frac{\delta}{\delta\bar{\psi}^f(x)}\,\mathrm{e}^{i(S[A_\mu,\psi,\bar{\psi}]+\int d^4x\,[\bar{\psi}^g\xi^g+\bar{\xi}^g\psi^g+A_\mu J^\mu])} \\
&= \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar{\psi}]\left\{\frac{\delta S}{\delta\bar{\psi}^f(x)}+\xi^f(x)\right\} \\
&\quad\times\exp\left\{i(S[A_\mu,\psi,\bar{\psi}]+\int d^4x\,[\bar{\psi}^g\xi^g+\bar{\xi}^g\psi^g+A_\mu J^\mu])\right\} \\
&= \left\{\frac{\delta S}{\delta\bar{\psi}^f(x)}\left[\frac{\delta}{i\delta J},\frac{\delta}{i\delta\bar{\xi}},-\frac{\delta}{i\delta\xi}\right]+\eta^f(x)\right\}W[J_\mu,\xi,\bar{\xi}] && (91) \\
0 &= \left[\xi^f(x)+\left(i\partial\!\!\!/-m_0^f+e_0^f\gamma^\mu\,\frac{\delta}{i\delta J^\mu(x)}\right)\frac{\delta}{i\delta\bar{\xi}^f(x)}\right]W[J_\mu,\xi,\bar{\xi}]. && (92)
\end{aligned}
$$

The last line furnishes a nonperturbative functional equivalent of Dirac's equation.

Functional Green Function

Next step … a functional derivative with respect to ξ^f: $\delta/\delta\xi^f(y)$, yields

$$\delta^4(x-y)W[J_\mu] - (i\not\partial - m_0^f + e_0^f\gamma^\mu \frac{\delta}{i\delta J^\mu(x)})W[J_\mu]\, S^f(x,y;[A_\mu]) = 0\,, \quad (93)$$

after setting $\xi^f = 0 = \bar\xi^f$, where $W[J_\mu] := W[J_\mu, 0, 0]$ and $S(x,y;[A_\mu])$ is defined in Eq. (76).

Now, using Eqs. (64), (71), this can be rewritten

$$\delta^4(x-y) - [i\not\partial - m_0^f + e_0^f \not A(x;[J]) + e_0^f\gamma^\mu \frac{\delta}{i\delta J^\mu(x)}]\, S^f(x,y;[A_\mu]) = 0\,, \quad (94)$$

which defines the nonperturbative connected 2-point fermion Green function

- NB. This is clearly the functional equivalent of Eq. (61):

$$[i\not\partial_{x'} - e\not A(x') - m]\, S(x',x) = 1\,\delta^4(x'-x)\,. \quad (95)$$

namely, Differential Operator Green Function for the Interacting Dirac Theory.

DSE for Fermion Propagator

The electromagentic four-potential vanishes in the absence of an external source; i.e., $A_\mu(x;[J=0])=0$

Remains only to exhibit the content of the remaining functional differentiation in Eq. (94), which can be accomplished using Eq. (81):

$$\frac{\delta}{i\delta J^\mu(x)}\, S^f(x,y;[A_\mu]) = \int d^4z\, \frac{\delta A_\nu(z)}{i\delta J^\mu(x)}\, \frac{\delta}{\delta A_\nu(z)} \left(\left. \frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar{\psi}^f(y)}\right|_{\psi=\overline{\psi}=0} \right)^{-1}$$

$$= \quad -e_0^f \int d^4z\, d^4u\, d^4w\, \frac{\delta A_\nu(z)}{i\delta J_\mu(x)}\, S^f(x,u)\, \Gamma_\nu(u,w;z)\, S(w,y)$$

$$= \quad -e_0^f \int d^4z\, d^4u\, d^4w\; iD_{\mu\nu}(x-z)\, S^f(x,u)\, \Gamma_\nu(u,w;z)\, S(w,y)\,, \qquad (96)$$

In the last line, we have set $J=0$ and used Eq. (80).

Hence in the absence of external sources Eq. (94) is equivalent to

$$\delta^4(x-y) = (i\partial\!\!\!/ - m_0^f)\, S^f(x,y)$$
$$- i\,(e_0^f)^2 \int d^4z\, d^4u\, d^4w\, D^{\mu\nu}(x,z)\, \gamma_\mu\, S(x,u)\, \Gamma_\nu(u,w;z)\, S(w,y)\,. \qquad (97)$$

Fermion Self-Energy

Photon vacuum polarisation was introduced to re-express the DSE for the gauge boson propagator, Eq. (85). Analogue, one can define a fermion self-energy:

$$\Sigma^f(x,z) = i(e_0^f)^2 \int d^4u\, d^4w\; D^{\mu\nu}(x,z)\, \gamma_\mu\, S(x,u)\, \Gamma_\nu(u,w;z)\,, \qquad (98)$$

so that Eq. (97) assumes the form

$$\int d^4z\, [(i\not\partial_x - m_0^f)\delta^4(x-z) - \Sigma^f(x,z)]\, S(z,y) = \delta^4(x-y)\,. \qquad (99)$$

Using property that Green functions are translationally invariant in the absence of external sources:

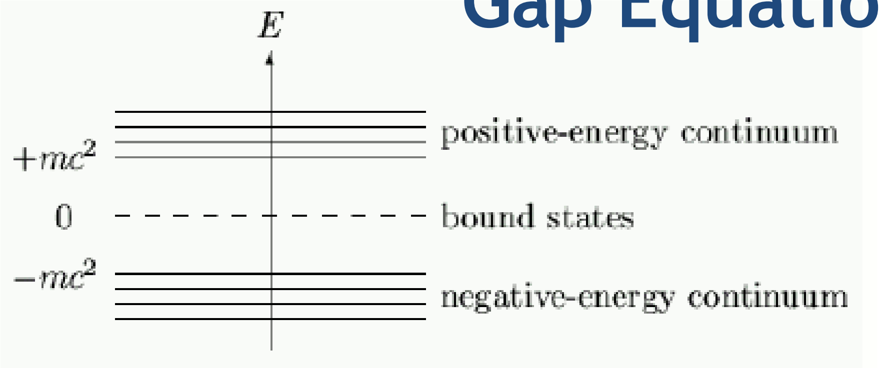
$$-i\Sigma^f(p) = (e_0^f)^2 \int \frac{d^4\ell}{(2\pi)^4}\, [iD^{\mu\nu}(p-\ell)]\, [i\gamma_\mu]\, [iS^f(\ell)]\, [i\Gamma_\nu^f(\ell,p)]\,. \qquad (100)$$

Now follows from Eq. (99) that connected fermion 2-point function in momentum space is

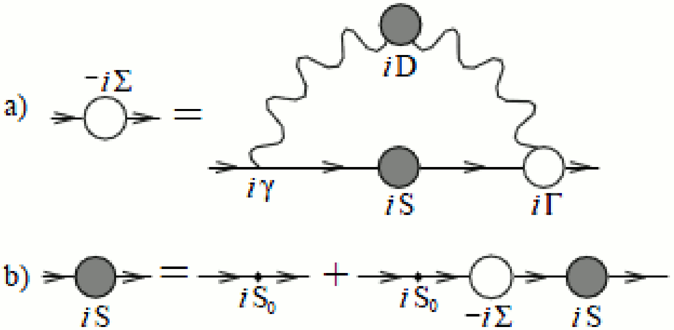
$$S^f(p) = \frac{1}{\not p - m_0^f - \Sigma^f(p) + i\eta^+}\,. \qquad (101)$$

Gap Equation

Equation (100) is the **exact** *Gap Equation*.



Describes manner in which propagation characteristics of a fermion moving through ground state of QED (the QED vacuum) is altered by the repeated emission and reabsorption of virtual photons.



- Equation can also describe the real process of Bremsstrahlung. Furthermore, solution of analogous equation in QCD provides information about dynamical chiral symmetry breaking and also quark confinement.

Perturbative Calculation of Gap

Keystone of strong interaction physics is dynamical chiral symmetry breaking (DCSB). In order to understand DCSB one must first come to terms with explicit chiral symmetry breaking. Consider then the DSE for the quark self-energy in QCD:

$$-i\,\Sigma(p) = -g_0^2 \int \frac{d^4\ell}{(2\pi)^4}\, D^{\mu\nu}(p-\ell)\, \frac{i}{2}\lambda^a\gamma_\mu\, S(\ell)\, i\Gamma^a_\nu(\ell,p)\,, \qquad (102)$$

where the flavour label is suppressed.

Form is precisely the same as that in QED, Eq. (100) but . . .

- colour (Gell-Mann) matrices: $\{\lambda^a; a = 1,\ldots,8\}$ at the fermion-gauge-boson vertex
- $D^{\mu\nu}(\ell)$ is the connected gluon 2-point function
- $\Gamma^a_\nu(\ell,\ell')$ is the proper quark-gluon vertex

One-loop contribution to quark's self-energy obtained by evaluating r.h.s. of Eq. (102) using the free quark and gluon propagators, and the quark-gluon vertex:

$$\Gamma^{a\,(0)}_\nu(\ell,\ell') = \frac{1}{2}\lambda^a\gamma_\nu\,. \qquad (103)$$

Explicit Leading-Order Computation

$$-i\,\Sigma^{(2)}(p) \;=\; -g_0^2 \int \frac{d^4k}{(2\pi)^4}\left[-g^{\mu\nu} + (1-\lambda_0)\frac{k^\mu k^\nu}{k^2+i\eta^+}\right]\frac{1}{k^2+i\eta^+} \times \frac{i}{2}\lambda^a\gamma_\mu\,\frac{1}{\not k + \not p - m_0 + i\eta^+}\,\frac{i}{2}\lambda^a\gamma_\mu\,. \quad (104)$$

To proceed, first observe that Eq. (104) can be re-expressed as

$$-i\,\Sigma^{(2)}(p) = -g_0^2\,C_2(R)\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k+p)^2-m_0^2+i\eta^+}\,\frac{1}{k^2+i\eta^+} \times\left\{\gamma^\mu\,(\not k+\not p+m_0)\,\gamma_\mu-(1-\lambda_0)\,(\not k-\not p+m_0)-2\,(1-\lambda_0)\,\frac{(k,p)\,\not k}{k^2+i\eta^+}\right\}, \quad (105)$$

where we have used $\frac{1}{2}\lambda^a\,\frac{1}{2}\lambda^a = C_2(R)\,\boldsymbol{I}_c\,;\; C_2(R) = \frac{N_c^2-1}{2N_c}$, with N_c the number of colours ($N_c = 3$ in QCD), and $\boldsymbol{I}_c$ is the identity matrix in colour space.

Explicit Leading-Order Computation

Now note that $2\,(k,p) = [(k+p)^2 - m_0^2] - [k^2] - [p^2 - m_0^2]$ and hence

$$\begin{aligned}
-\,i\,\Sigma^{(2)}(p) \;=\; & -g_0^2\,C_2(R)\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{(k+p)^2 - m_0^2 + i\eta^+}\,\frac{1}{k^2 + i\eta^+} \\
& \Big\{ \gamma^\mu\,(\not k + \not p + m_0)\,\gamma_\mu + (1-\lambda_0)\,(\not p - m_0) \\
& + (1-\lambda_0)\,(p^2 - m_0^2)\,\frac{\not k}{k^2 + i\eta^+} \\
& - (1-\lambda_0)\,[(k+p)^2 - m_0^2]\,\frac{\not k}{k^2 + i\eta^+} \Big\}. \qquad (106)
\end{aligned}$$

Focus on the last term:

$$\begin{aligned}
& \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{(k+p)^2 - m_0^2 + i\eta^+}\,\frac{1}{k^2 + i\eta^+}\,[(k+p)^2 - m_0^2]\,\frac{\not k}{k^2 + i\eta^+} \\
= \;& \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2 + i\eta^+}\,\frac{\not k}{k^2 + i\eta^+} = 0 \qquad (107)
\end{aligned}$$

because the integrand is odd under $k \to -k$, and so this term in Eq. (106) vanishes.

Explicit Leading-Order Computation

$$-i\,\Sigma^{(2)}(p) = -g_0^2\,C_2(R)\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k+p)^2-m_0^2+i\eta^+}\,\frac{1}{k^2+i\eta^+}$$
$$\left\{\gamma^\mu\,(\not k+\not p+m_0)\,\gamma_\mu+(1-\lambda_0)\,(\not p-m_0)+(1-\lambda_0)\,(p^2-m_0^2)\,\frac{\not k}{k^2+i\eta^+}\right\}.$$

Consider the second term:

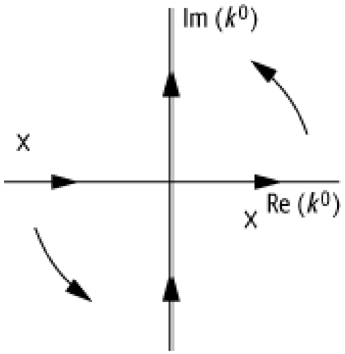
$$(1-\lambda_0)\,(\not p-m_0)\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k+p)^2-m_0^2+i\eta^+}\,\frac{1}{k^2+i\eta^+}\,.$$

In particular, focus on the behaviour of the integrand at large k^2:

$$\frac{1}{(k+p)^2-m_0^2+i\eta^+}\,\frac{1}{k^2+i\eta^+}\overset{k^2\to\pm\infty}{\sim}\frac{1}{(k^2-m_0^2+i\eta^+)\,(k^2+i\eta^+)}\,. \qquad (108)$$

Wick Rotation

Integrand has poles in the second and fourth quadrant of the complex-k_0-plane but vanishes on any circle of radius $R \to \infty$ in this plane. That means one may rotate the contour anticlockwise to find



$$\begin{aligned} \int_0^\infty dk^0 \, \frac{1}{(k^2 - m_0^2 + i\eta^+)\,(k^2 + i\eta^+)} \\ = \int_0^{i\infty} dk^0 \, \frac{1}{([k^0]^2 - \vec{k}^2 - m_0^2 + i\eta^+)([k^0]^2 - \vec{k}^2 + i\eta^+)} \\ \overset{k^0 \to ik_4}{=} \; i \int_0^\infty dk_4 \, \frac{1}{(-k_4^2 - \vec{k}^2 - m_0^2)\,(-k_4^2 - \vec{k}^2)} \,. \end{aligned} \tag{109}$$

Performing a similar analysis of the $\int_{-\infty}^0$ part, one obtains the complete result:

$$\begin{aligned} \int \frac{d^4k}{(2\pi)^4} \, \frac{1}{(k^2 - m_0^2 + i\eta^+)\,(k^2 + i\eta^+)} \\ = \; i \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dk_4}{2\pi} \, \frac{1}{(-\vec{k}^2 - k_4^2 - m_0^2)\,(-\vec{k}^2 - k_4^2)} \,. \end{aligned} \tag{110}$$

These two steps constitute what is called a *Wick rotation*.

Euclidean Integral

The integral on the r.h.s. is defined in a four-dimensional Euclidean space; i.e., $k^2 := k_1^2 + k_2^2 + k_3^2 + k_4^2 \geq 0$, with k^2 nonnegative.

A general vector in this space can be written in the form:

$$(k) = |k| \left(\cos\phi \; \sin\theta \; \sin\beta, \sin\phi \; \sin\theta \; \sin\beta, \cos\theta \; \sin\beta, \cos\beta\right); \qquad (111)$$

i.e., using hyperspherical coordinates, and clearly $k^2 = |k|^2$.

In this Euclidean space using these coordinates the four-vector measure factor is

$$\int d_E^4 k \, f(k_1, \ldots, k_4) = \frac{1}{2}\int_0^\infty dk^2 \, k^2 \int_0^\pi d\beta \, sin^2\beta \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \, f(k, \beta, \theta, \phi) \, . \qquad (112)$$

Euclidean Integral

Returning to Eq. (108) and making use of the material just introduced, the large k^2 behaviour of the integral can be determined via

$$
\begin{aligned}
&\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k+p)^2 - m_0^2 + i\eta^+} \frac{1}{k^2 + i\eta^+} \\
&\approx \quad \frac{i}{16\pi^2} \int_0^\infty dk^2 \frac{1}{(k^2 + m_0^2)} \\
&= \quad \frac{i}{16\pi^2} \lim_{\Lambda\to\infty} \int_0^{\Lambda^2} dx \frac{1}{x + m_0^2} \\
&= \quad \frac{i}{16\pi^2} \lim_{\Lambda\to\infty} \ln(1 + \Lambda^2/m_0^2) \to \infty\,; \qquad (113)
\end{aligned}
$$

After all this work, the result is meaningless: the one-loop contribution to the quark's self-energy is divergent!

Regularisation and Renormalisation

- Such "ultraviolet" divergences, and others which are more complicated, arise whenever loops appear in perturbation theory. (The others include "infrared" divergences associated with the gluons' masslessness; e.g., consider what would happen in Eq. (113) with $m_0 \to 0$.)
- In a *renormalisable* quantum field theory there exists a well-defined set of rules that can be used to render perturbation theory sensible.
  - First, however, one must *regularise* the theory; i.e., introduce a cutoff, or use some other means, to make finite every integral that appears. Then each step in the calculation of an observable is rigorously sensible.
  - *Renormalisation* follows; i.e, the absorption of divergences, and the redefinition of couplings and masses, so that finally one arrives at S-matrix amplitudes that are finite and physically meaningful.
- The regularisation procedure must preserve the Ward-Takahashi identities (the Slavnov-Taylor identities in QCD) because they are crucial in proving that a theory can sensibly be renormalised.
- A theory is called renormalisable if, and only if, number of different types of divergent integral is finite. Then only finite number of masses & couplings need to be renormalised; i.e., *a priori* the theory has only a finite number of undetermined parameters that must be fixed through comparison with experiments.

Renormalised One-Loop Result

Don't have time to explain and illustrate the procedure. Interested?
Read . . . Pascual, P. and Tarrach, R. (1984), Lecture Notes in Physics, Vol. **194**, *QCD: Renormalization for the Practitioner* (Springer-Verlag, Berlin).

Answer, in Momentum Subtraction Scheme:

$$\Sigma^{(2)}_R(\not p) = \Sigma^{(2)}_{VR}(p^2)\,\not p + \Sigma^{(2)}_{SR}(p^2)\,1_D;$$

$$\Sigma^{(2)}_{VR}(p^2;\zeta^2) = \frac{\alpha(\zeta)}{\pi}\,\lambda(\zeta)\,\frac{1}{4}\,C_2(R)\left\{ -m^2(\zeta)\left(\frac{1}{p^2}+\frac{1}{\zeta^2}\right) + \left(1-\frac{m^4(\zeta)}{p^4}\right)\ln\left(1-\frac{p^2}{m(\zeta)^2}\right) - \left(1-\frac{m^4(\zeta)}{\zeta^4}\right)\ln\left(1+\frac{\zeta^2}{m^2(\zeta)}\right)\right\},$$

$$\Sigma^{(2)}_{SR}(p^2;\zeta^2) = m(\zeta)\,\frac{\alpha(\zeta)}{\pi}\,\frac{1}{4}\,C_2(R)\left\{ -[3+\lambda(\zeta)] \times\left[\left(1-\frac{m^2(\zeta)}{p^2}\right)\ln\left(1-\frac{p^2}{m^2(\zeta)}\right) - \left(1+\frac{m^2(\zeta)}{\zeta^2}\right)\ln\left(1+\frac{\zeta^2}{m^2(\zeta)}\right)\right]\right\},$$

where the renormalised quantities depend on the point at which the renormalisation has been conducted;
e.g., $\alpha(\zeta)$ is the running coupling, $m(\zeta)$ is the running quark mass.

Observations on perturbative quark self-energy

- QCD is Asymptotically Free. Hence, at some large spacelike $p^2 = \zeta^2$ the propagator is exactly the free propagator *except* that the bare mass is replaced by the renormalised mass.
- At one-loop order, the vector part of the dressed self energy is proportional to the running gauge parameter. In Landau gauge, that parameter is zero. Hence, the vector part of the renormalised dressed self energy vanishes at one-loop order in perturbation theory.
- The scalar part of the dressed self energy is proportional to the renormalised current-quark mass.
  - This is true at one-loop order, and at every order in perturbation theory.
  - Hence, if current-quark mass vanishes, then $\Sigma_R \equiv 0$ in perturbation theory. That means if one starts with a chirally symmetric theory, one ends up with a chirally symmetric theory.

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No Dynamical Chiral Symmetry Breaking in perturbation theory

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Overarching Science Questions for the coming decade: 2013-2022

- Discover meaning of confinement;
- its relationship to DCSB;
- and the nature of the transition between the nonperturbative & perturbative domains of QCD

… coming lectures