# Hadron Phenomenology and QCDs DSEs

## Lecture 3: *Relativistic Scattering and Bound State Equations*

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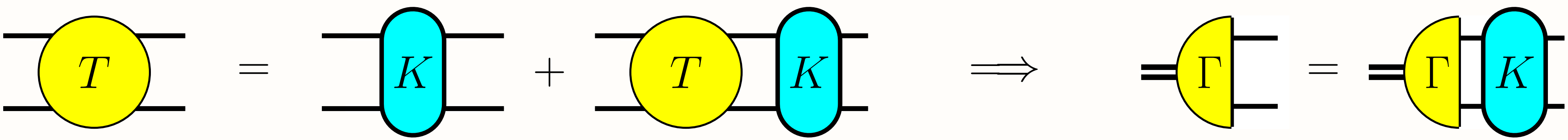
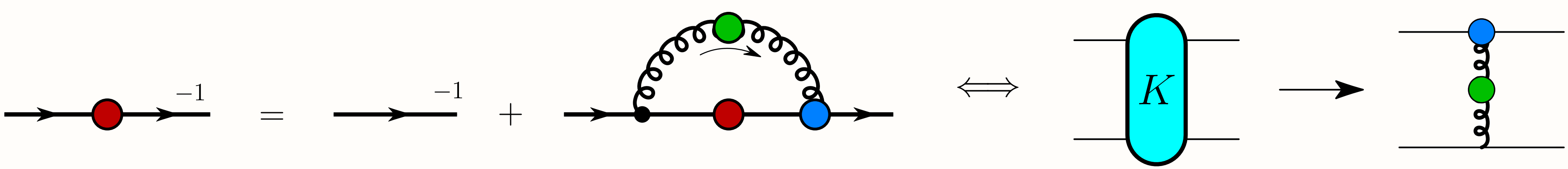
# *Hadron Spectrum*

- In quantum field theory physical states appear as poles in $n$-point Green Functions
- For example, the quark–antiquark scattering matrix or $t$-matrix, contains poles for all $\bar{q}q$ bound states, that is, the physical mesons
- The quark–antiquark $t$-matrix is obtained by solving the Bethe-Salpeter equation



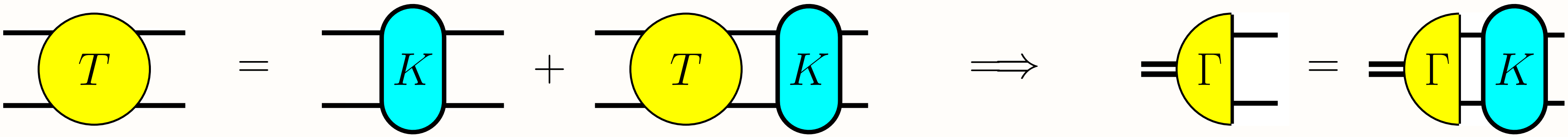

- Kernels of gap and BSE must be intimately related




- For example, consider axial–vector Ward–Takahashi Identity

$$q_\mu \, \Gamma_5^{\mu,i}(p',p) = S^{-1}(p') \, \gamma_5 \, \tfrac{1}{2}\tau_i + \tfrac{1}{2}\tau_i \, \gamma_5 \, S^{-1}(p) + 2\, m \, \Gamma_\pi^i(p',p)$$

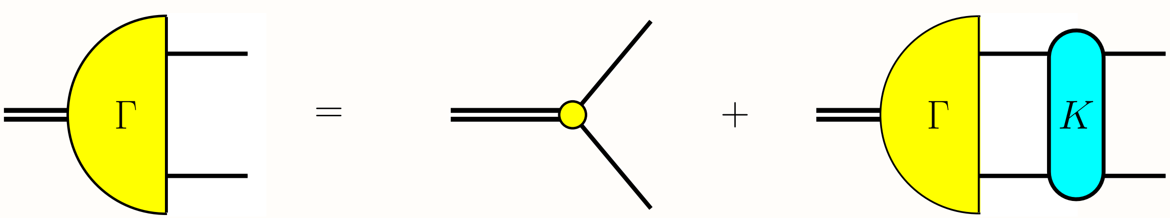
# *Inhomogeneous & Homogeneous vertex functions*




- Near a bound state pole of mass $m$ a two-body $t$-matrix behaves as

$$\mathcal{T}(p,k) \to \frac{\Gamma(p,k)\bar{\Gamma}(p,k)}{p^2 - m^2} \qquad \text{where} \qquad p = p_1 + p_2, \quad k = p_1 - p_2$$

- $\Gamma(p,k)$ is the homogeneous Bethe-Salpeter vertex and describes the relative motion of the quark and anti-quark while they form the bound state
- The inhomogeneous BSE is a generalization and contains all the poles of the $t$-matrix

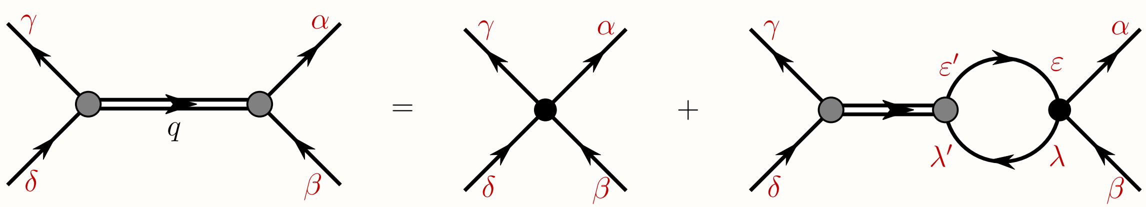



  - ✦ the first term on the RHS is the elementary driving term
  - ✦ the driving term projects out various channels
  - ✦ the quark-photon vertex is described by a inhomogeneous BSE

# The Pion

- How does the pion become (almost) massless when it is composed of two massive constituents
- The pion is realized as the lowest lying pole in the quark anti-quark $t$-matrix in the pseudoscalar channel
- In the NJL model this $t$-matrix is given by

$$\mathcal{T}(q)_{\alpha\beta,\gamma\delta} = \mathcal{K}_{\alpha\beta,\gamma\delta} + \int \frac{d^4k}{(2\pi)^4}\, \mathcal{K}_{\alpha\beta,\lambda\epsilon}\, S(q+k)_{\varepsilon\varepsilon'}\, S(k)_{\lambda'\lambda}\, \mathcal{T}(q)_{\varepsilon'\lambda',\gamma\delta},$$



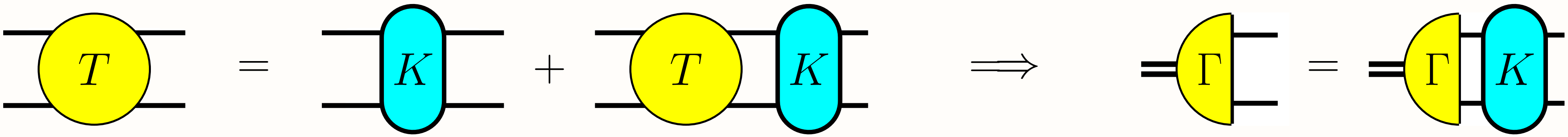

$$\mathcal{K} = -2i\, G_\pi\, (\gamma_5 \boldsymbol{\tau})_{\alpha\beta} (\gamma_5 \boldsymbol{\tau})_{\lambda\epsilon}$$

- The NJL pion $t$-matrix is

$$\mathcal{T}(q)^i_{\alpha\beta,\gamma\delta} = (\gamma_5 \tau_i)_{\alpha\beta}\, \frac{-2i\, G_\pi}{1 + 2\, G_\pi\, \Pi_{PP}(q^2)}\, (\gamma_5 \tau_i)_{\lambda\epsilon}$$

- The pion mass is then given by: $1 + 2\, G_\pi\, \Pi_{PP}(q^2 = m_\pi^2) = 0$

# Pion Bethe-Salpeter Amplitude




- Recall that near a bound state pole the $t$-matrix behaves as

$$\mathcal{T}(p,k) \to \frac{\Gamma(p,k)\bar{\Gamma}(p,k)}{p^2 - m^2} \qquad \text{where} \qquad p = p_1 + p_2, \quad k = p_1 - p_2$$

- Expanding the pion $t$-matrix about the pole gives

$$\mathcal{T}(q) = (\gamma_5 \tau_i)\, \frac{-2i\, G_\pi}{1+2\, G_\pi\, \Pi_{PP}(q^2)}\, (\gamma_5 \tau_i) \to \frac{i\, g_\pi}{q^2 - m_\pi^2} (\gamma_5 \tau_i)(\gamma_5 \tau_i)$$

- Where $g_\pi$ is interpreted as the pion-quark coupling constant

$$g_\pi = -\frac{1}{\frac{\partial}{\partial q^2}\, \Pi_{PP}(q^2)}\Bigg|_{q^2 = m_\pi^2}$$

- The pion homogeneous BS vertex is therefore: $\Gamma_\pi = \sqrt{g_\pi}\, \gamma_5 \tau_i$
  - ✦ this is a very simple vertex that misses a lot of physics

# *The Pion as a Goldstone Boson*

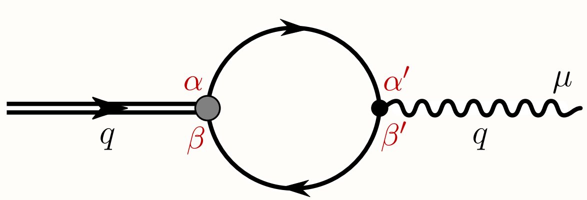
- Using the NJL gap equation and the pion pole condition gives

$$m_\pi^2 = m\,\frac{2\,M}{G_\pi\,\Pi_{AA}^{(L)}(m_\pi^2)}$$

- Therefore in the chiral limit, $m \to 0$, the pion is massless
- In quantum mechanics one could tune a potential to give a massless ground state for a bound state of two massive constituents
  - ✦ however quantum mechanics always gives: $M_{\text{bound state}} \propto M_{\text{constituents}}$
- However quantum field theory with DCSB gives: $m_\pi^2 \propto m$
- Recall the Gell-Mann–Oakes–Renner relation

$$f_\pi^2\,m_\pi^2 = \tfrac{1}{2}\,(m_u + m_d)\,\langle \bar{u}u + \bar{d}d\rangle$$

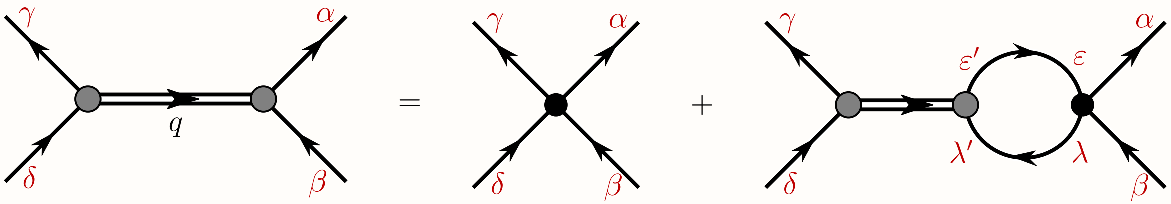
- The pion decay constant is given by




$$\langle 0\,|A_a^\mu|\,\pi_b(p)\rangle = i\,f_\pi\,p^\mu\,\delta_{ab}$$

# $\rho$–$a_1$ mass splitting

- The $\rho$ and $a_1$ are the lowest lying vector ($J^P = 1^-$) and axial-vector ($J^P = 1^+$) $\bar{q}q$ bound states: $m_\rho \simeq 770$ MeV & $m_{a_1} \simeq 1230$ MeV
- The masses of these states are given by $t$-matrix poles in the vector and axial-vector $\bar{q}q$ channels




$$\mathcal{K} = -2i\, G_\rho \Big[ (\gamma_\mu \boldsymbol{\tau})_{\alpha\beta} (\gamma^\mu \boldsymbol{\tau})_{\gamma\delta} + (\gamma_\mu \gamma_5 \boldsymbol{\tau})_{\alpha\beta} (\gamma^\mu \gamma_5 \boldsymbol{\tau})_{\gamma\delta} \Big]$$

- Chiral symmetry implies one NJL coupling, $G_\rho$. NJL gives

$$m_\rho \equiv 770\,\text{MeV} \;\; \& \;\; m_{a_1} \simeq 1000\,\text{MeV}$$

✦ NJL interaction is insufficient to obtain correct $\rho$–$a_1$ mass splitting

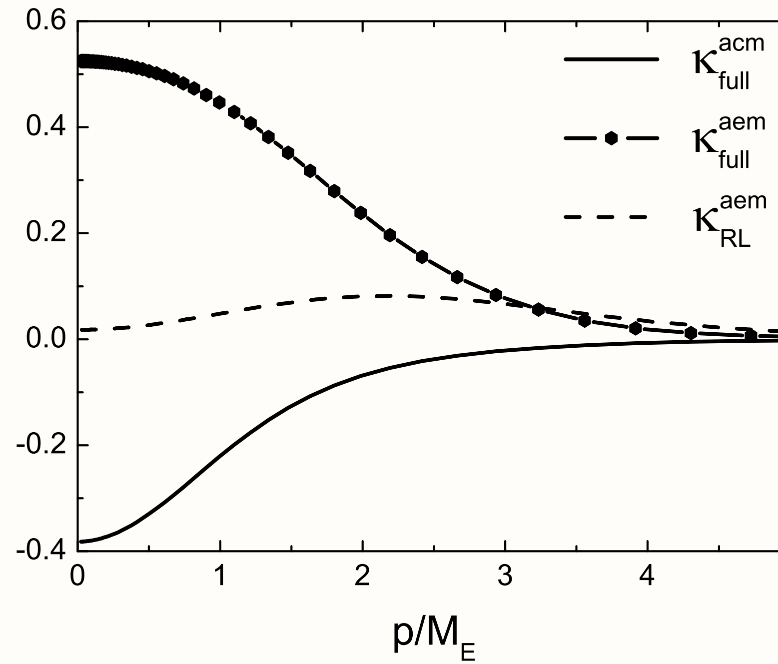
- The rainbow ladder Maris–Tandy DSE model gives

$$m_\rho \simeq 644\,\text{MeV} \;\; \& \;\; m_{a_1} \simeq 759\,\text{MeV}$$

- Clearly something is missing!

# $\rho$–$a_1$ mass splitting – Generalized Quark-Gluon Vertex

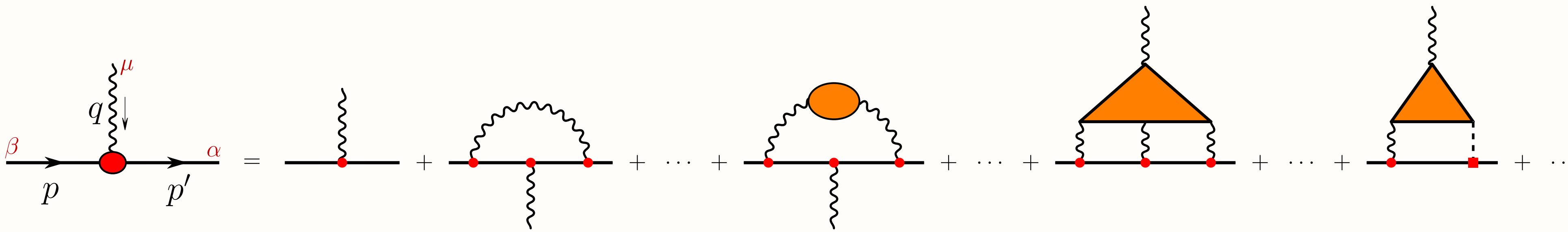
- Recall that going below rainbow ladder by adding $\sigma^{\mu\nu} q_\nu \, \tau_5(p', p)$ to quark-gluon vertex, generates quark anomalous magnetic moment
- An order parameter for dynamical chiral symmetry; Chiral symmetry forbids massless particle having anomalous magnetic moments
- What about the hadron spectrum



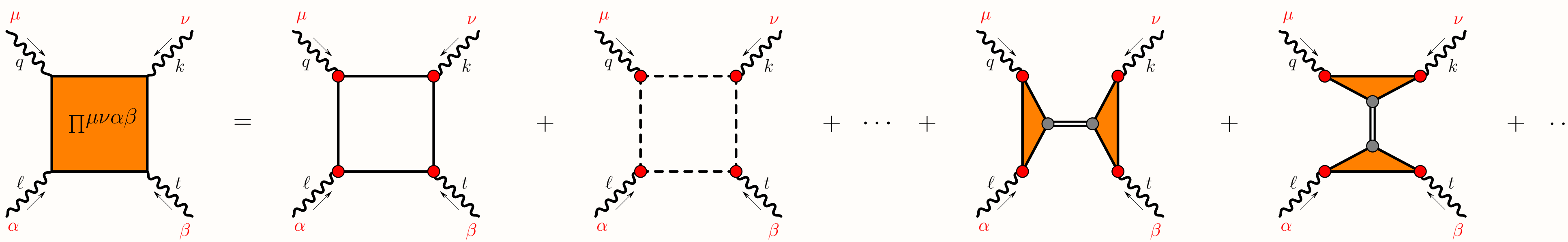

| | Experiment | Rainbow-ladder | One-loop corrected | Ball-Chiu | Full vertex |
|---|---|---|---|---|---|
| a1 | 1230 | 759 | 885 | 1128 | 1270 |
| ρ | 770 | 644 | 764 | 919 | 790 |
| Mass splitting | 455 | 115 | 121 | 209 | 480 |

- *Important example of interplay between observables and DSE quark–gluon vertex*

# An Aside – Muon Anomalous Magnetic Moment




- $a_\mu^{\text{exp}} = 11659208.0 \pm 6.3 \times 10^{-10}$; $a_\mu^{\text{theory}} = 11659179.0 \pm 6.5 \times 10^{-10}$
- largest theory error come from HLBL scattering contribution

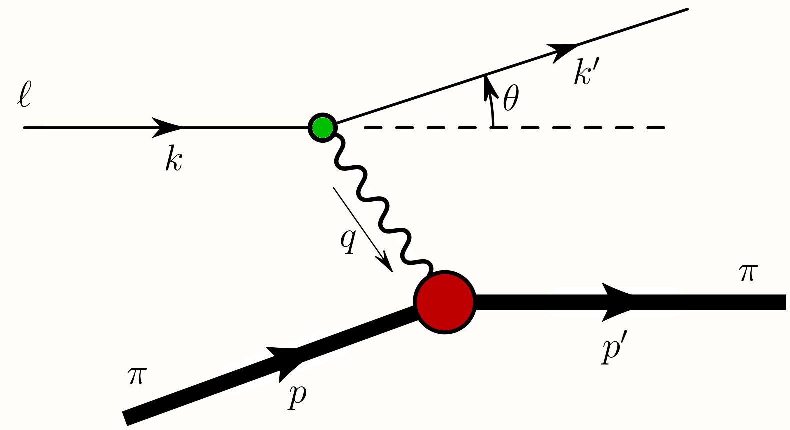



- Box diagram contribution is least know
  - ✦ only $\gamma^\mu$ coupling and VMD has been considered so far
  - ✦ we argue that the anomalous magnetic moment term cannot be ignored
- At least error on $a_\mu^{\text{HLBL}} = 8.3 \pm 3.2 \times 10^{-10}$ should be much larger
- Fred Jegerlehner, Andreas Nyffeler, Physics Reports 477 (2009) 1–110

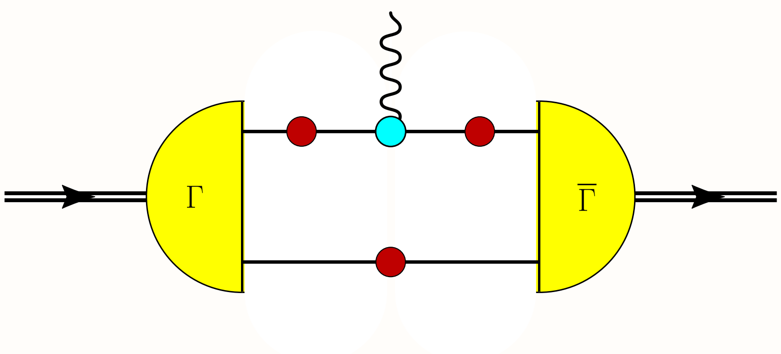
# Pion Form Factor

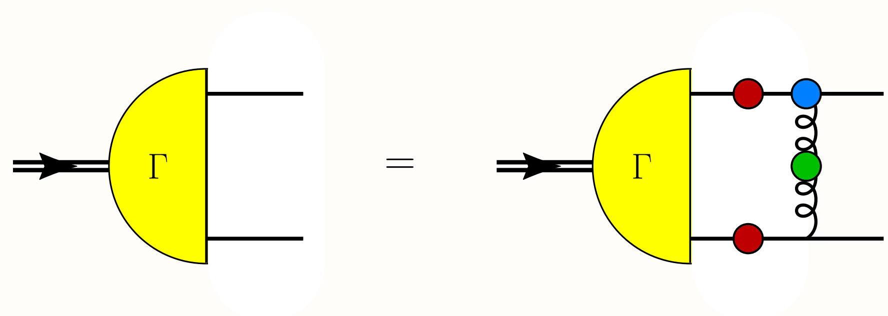
- Hadron form factors describe its interaction with the electromagnetic current
- An on-shell pion has one EM form factor

$$\langle J^\mu_\pi \rangle = (p'^\mu + p^\mu)\, F_1(Q^2)$$



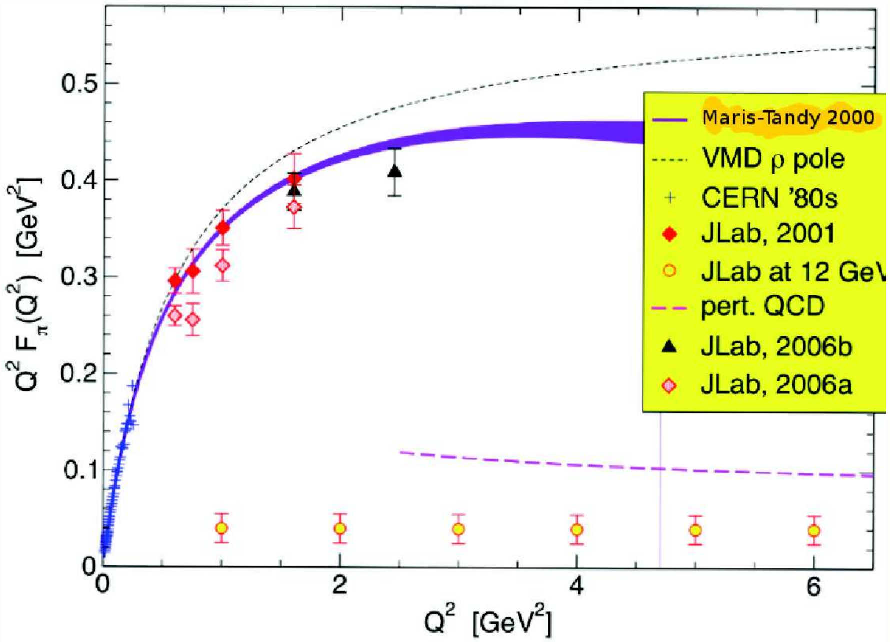

- In the impulse approximation the pion form factors are given by

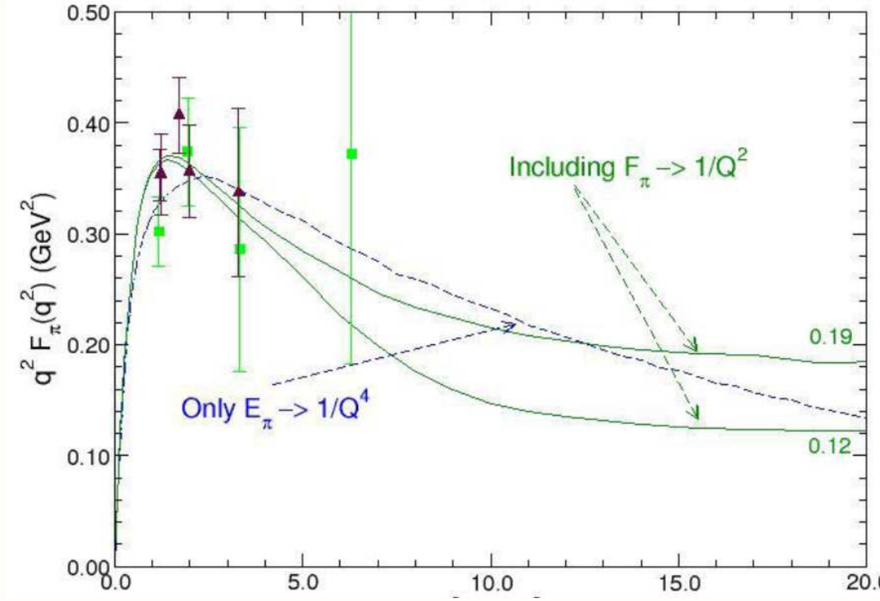



- Ingredients:
  - ✦ dressed quark propagators
  - ✦ homogeneous Bethe-Salpeter vertices
  - ✦ dressed quark-photon vertex

# Pion Form Factor (2)




- Pion BSE vertex has the general form

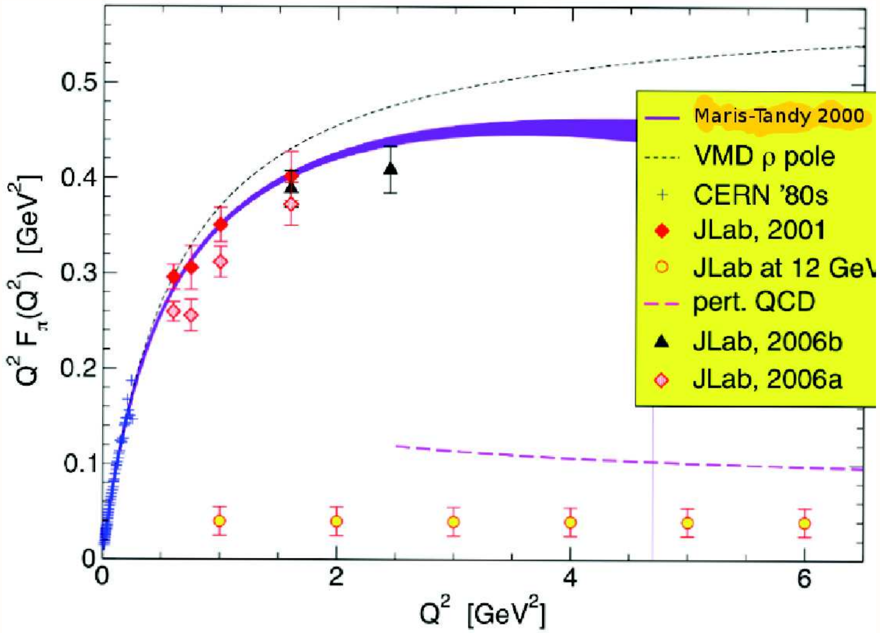
$$\Gamma_\pi(p,k) = \gamma_5 \Big[ E_\pi(p,k) + \not p\, F_\pi(p,k) + \not k\, k\cdot p\, \mathcal{G}(p,k) + \sigma^{\mu\nu} k_\mu p_\nu\, \mathcal{H}(p,k) \Big]$$

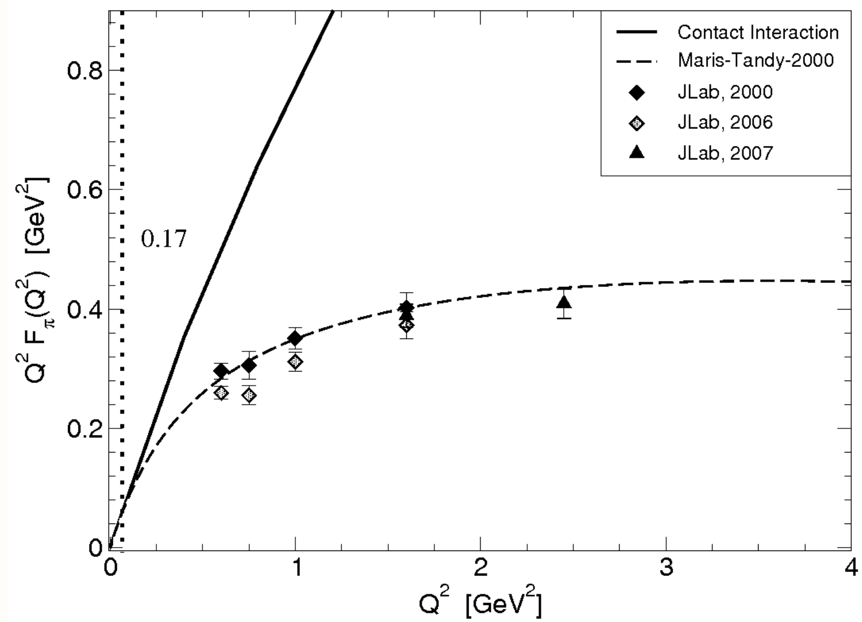
- Pseudovector component $F_\pi(p,k)$ dominates ultra-violet
- Perturbative QCD predicts

$$Q^2\, F_{1\pi}(Q^2) \overset{Q^2\to\infty}{=} 16\,\pi\, f_\pi^2\, \alpha_s(Q^2) \simeq 0.44\, \alpha_s(Q^2)$$

- DSE finds that pQCD sets in at about $Q^2 = 8\,\mathrm{GeV}^2$

# Some Consequences of Running Quark Mass




- Pion BSE vertex has the general form

$$\Gamma_\pi(p,k) = \gamma_5 \Big[ E_\pi(p,k) + \not{p}\, F_\pi(p,k) + \not{k}\, k\cdot p\, \mathcal{G}(p,k) + \sigma^{\mu\nu} k_\mu p_\nu\, \mathcal{H}(p,k) \Big]$$

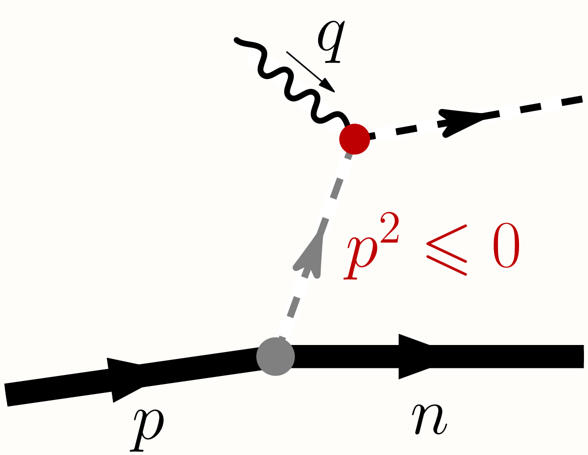
- In gap equation use simple kernel $\Longleftrightarrow$ NJL model with $\pi - a_1$ mixing

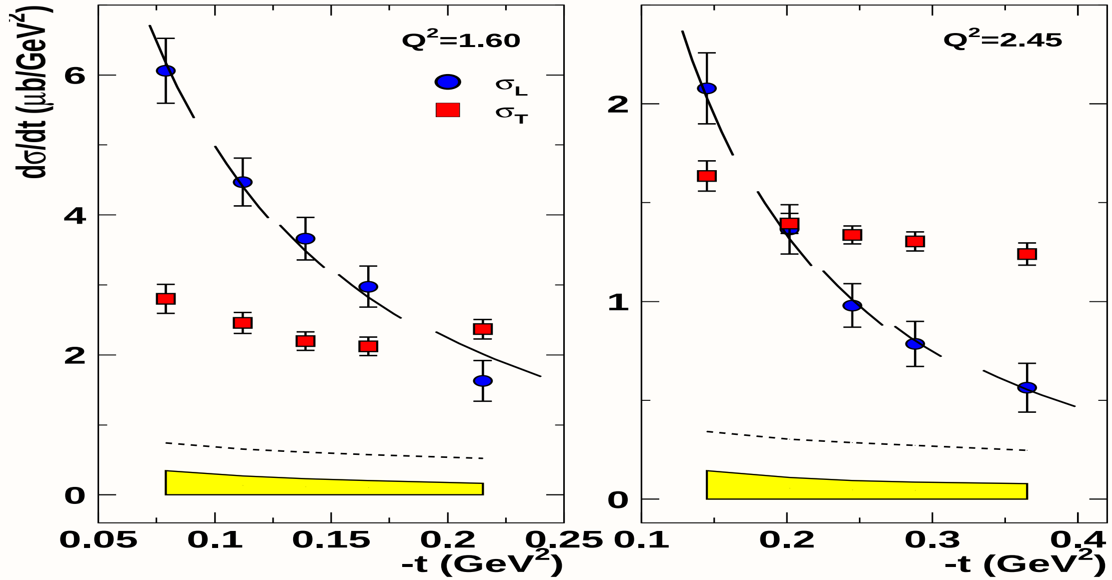
$$g^2 D_{\mu\nu}(p-k)\Gamma^\nu(p,k) \;\to\; \frac{1}{m_G^2}\, g_{\mu\nu}\,\gamma^\nu \;\Longrightarrow\; \Gamma_\pi(p,k) = \gamma_5 \left[E_\pi + \not{p}\, F_\pi\right]$$

  - ✦ Quark no longer has a running mass

- Nature of interaction has drastic observable consequences for $Q^2 > 0$

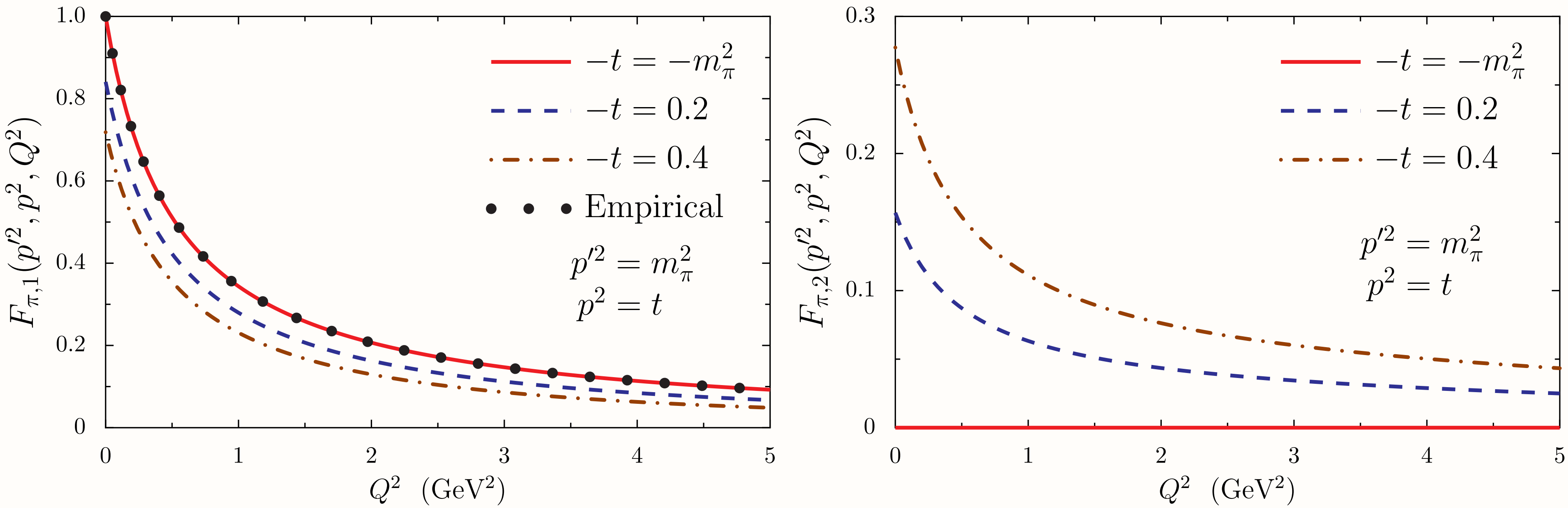
# *Measuring Pion Form Factor*




- At low $Q^2$ pion form factor is measured by scattering a pion from the electron cloud of an atom $[t \equiv p^2]$
  - ✦ small mass of electron limits this to $Q^2 < 0.5\,\text{GeV}^2$
- Higher $Q^2$ experiments have been performed at Jefferson Lab where a virtual photon scatters from a virtual pion that is part of the nucleon wavefunction
- Initial pion is off its mass shell – $p^2 \leqslant 0$ – on mass shell $p^2 = m_\pi^2$
  - ✦ need to extrapolate to the pion pole $p^2 = m_\pi^2$

# Off-Shell Pion Form Factors



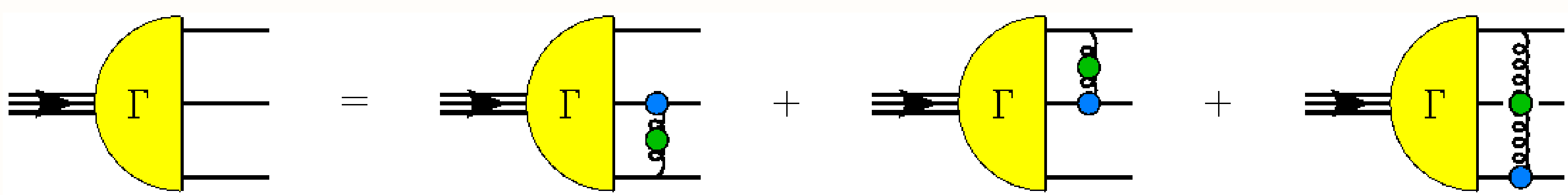

- Initial pion is off its mass shell – $p^2 < 0$ – on mass shell $p^2 = m_\pi^2$
  - ✦ need to extrapolate to the pion pole $p^2 = m_\pi^2$
- However an off-shell pion has two form factors not one and each form factor is a three dimensional function

$$\Gamma_\pi^\mu(p', p) = (p'^\mu + p^\mu)\, F_{\pi 1}(p'^2, p^2, q^2) + (p'^\mu - p^\mu)\, F_{\pi 2}(p'^2, p^2, q^2)$$

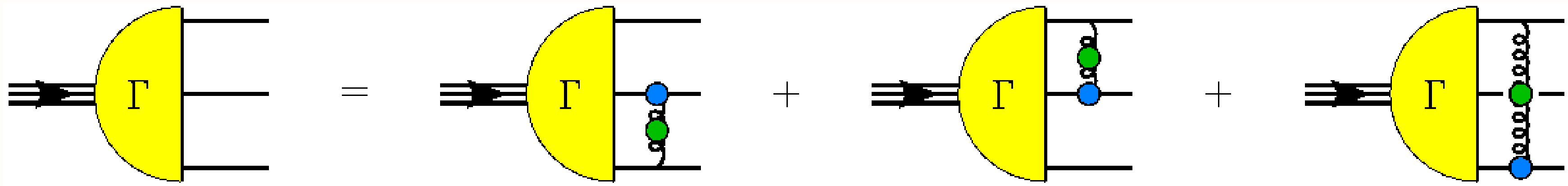
- Using NJL model can determine off-shell pion form factors
- Potentially important for experimental extraction of $F_\pi$

# *Baryons in the DSEs*

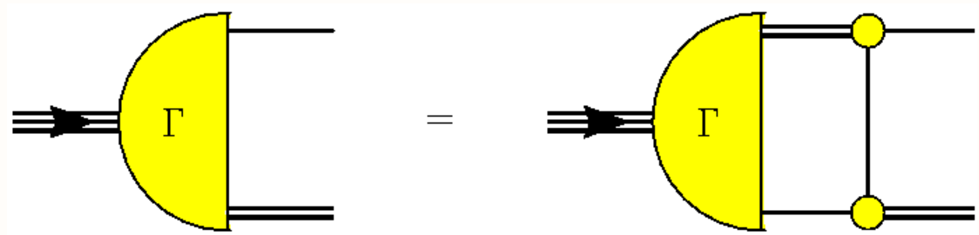
- Baryons are 3-quark bound states – with the proton $(uud)$ and neutron $(udd)$ being the most important examples
- In quantum field theory physical baryons appear as poles in six-point Green Functions
- Recall that two-body bound states appear as poles in four-point Green Functions and solutions were obtained by solving the Bethe-Salpeter Equation
- The analogue of the Bethe-Salpeter equation for 3-quark bound states is called the Faddeev equation
- By definition the Faddeev kernel only contains two-body interactions
  - this is an approximation which is yet to be explored and could have important consequences for QCD
- Diagrammatically the homogeneous Faddeev equation is given by



# *Baryons in the DSEs (2)*



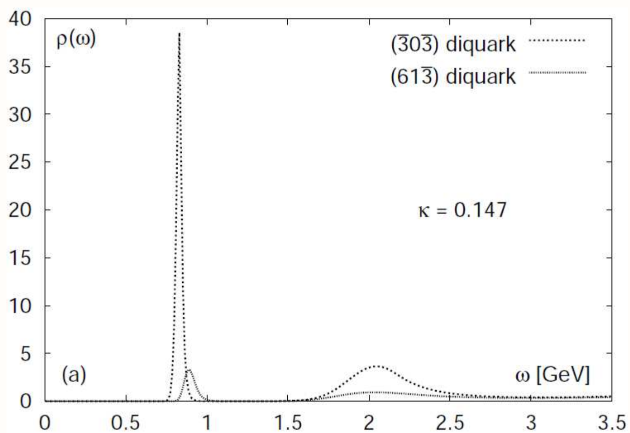

- We will render this problem tractable by making the quark-diquark approximation




- This is a linear matrix equation, whose solution gives the "Baryon wavefunction" – strictly the Poincaré covariant Faddeev amplitude
- We will include scalar ($J^P = 0^+, T = 0$) and axial-vector diquarks ($J^P = 1^+, T = 1$)
  - ✦ parity dictates that pseudoscalar and vector diquarks must be in an $\ell = 1$ state and are therefore suppressed in the nucleon
  - ✦ for the negative parity $N^*(1535)$ the opposite is true
- The nucleon wavefunction contains $S$, $P$ and $D$ wave correlations
- Equation has discrete solutions at $p^2 = m_i^2$;  nucleon, roper, etc

# What is a Diquark

- A diquark is a correlated (interacting) quark-quark state
- This interaction is attractive in the colour $\bar{3}$ (antisymmetric) or colour $6$ (symmetric), however only the colour $\bar{3}$ can exist inside a colour singlet nucleon
- Diquarks are analogous to mesons – colour singlet $\bar{q}q$ bound states
- Because diquarks are coloured they should not appear as physical states in QCD $\Longleftrightarrow$ confinement
- However in the rainbow ladder approximation and the NJL model diquarks do appear as poles in the $qq$ scattering $(t)$ matrix
- Lattice QCD also sees evidence for diquarks

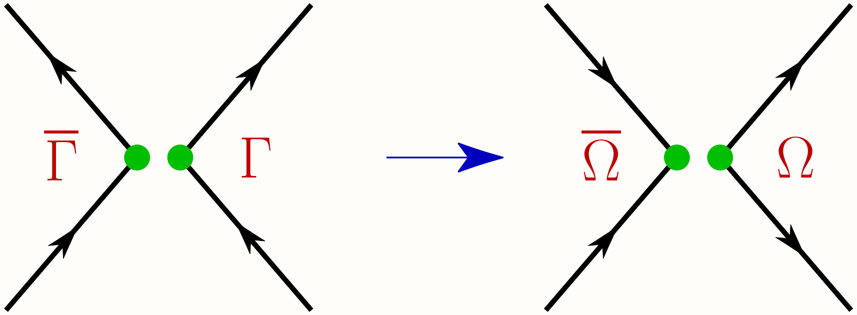



- I. Wetzorke, F. Karsch, hep-lat/0008008
- $(\bar{3}0\bar{3})$ implies scalar diquark: (flavour-$\bar{3}$, spin-$0$, colour-$\bar{3}$)
- $(60\bar{3})$ implies axial-vector diquark: (flavour-$6$, spin-$0$, colour-$\bar{3}$)

# Diquarks in The NJL model

- To describe diquarks in the NJL model one usually rewrites the $\bar{q}q$ interaction Lagrangian into a $qq$ interaction Lagrangian

$$\left(\bar{\psi}\,\Gamma\,\psi\right)^2 \to \left(\bar{\psi}\,\Omega\,\bar{\psi}^T\right)\left(\psi^T\,\bar{\Omega}\,\psi\right)$$




  - ✦ $\Omega$ has quantum numbers if interaction channel

- The $qq$ NJL Lagrangian in the scalar and axial-vector diquark channels has the form

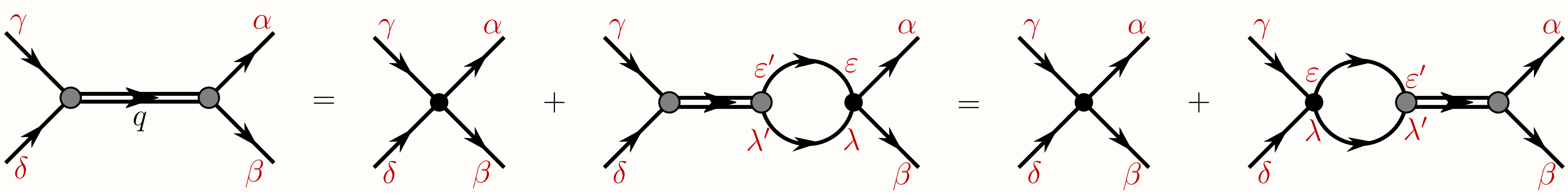
$$\mathcal{L}_I = G_s\left[\overline{\psi}\,\gamma_5\,C\,\tau_2\,\beta^A\,\overline{\psi}^T\right]\left[\psi^T\,C^{-1}\,\gamma_5\,\tau_2\,\beta^{A'}\,\psi\right] + G_a\left[\overline{\psi}\,\gamma_\mu\,C\,\tau_i\tau_2\,\beta^A\,\overline{\psi}^T\right]\left[\psi^T\,C^{-1}\,\gamma^\mu\,\tau_2\tau_j\,\beta^{A'}\,\psi\right] + \ldots.$$

  - ✦ the first term is the scalar diquark channel ($J^P = 0^+, T = 0$)
  - ✦ the second the axial-vector diquark channel ($J^P = 1^+, T = 1$)
  - ✦ $\tau_2$ couples isospin of two quarks to $T = 0$, $C\gamma_5$ couples spin to $J = 0$, $\beta^A = \sqrt{\frac{3}{2}}\,\lambda^A \quad A = 2, 5, 7$

# NJL diquark $t$-matrices

- The equation for the $qq$ scattering matrix – the Bethe-Salpeter equation has the form

$$\mathcal{T}(q)_{\alpha\beta,\gamma\delta} = K_{\alpha\beta,\gamma\delta} + \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, K_{\alpha\beta,\varepsilon\lambda}\, S(k)_{\varepsilon\varepsilon'}\, S(q-k)_{\lambda\lambda'}\, \mathcal{T}(q)_{\varepsilon'\lambda',\gamma\delta},$$




- ✦ note symmetry factor of $\frac{1}{2}$ (c.f. $\bar{q}q$ BSE)

- The Feynman rules for the interaction kernels are

$$\mathcal{K}_s = 4i\,G_s\left(\gamma_5\, C\,\tau_2\,\beta^A\right)_{\alpha\beta}\left(C^{-1}\,\gamma_5\,\tau_2\,\beta^A\right)_{\gamma\delta} \qquad \mathcal{K}_a = 4i\,G_a\left(\gamma_\mu\, C\,\tau_i\tau_2\,\beta^A\right)_{\alpha\beta}\left(C^{-1}\,\gamma^\mu\,\tau_2\tau_i\,\beta^A\right)_{\gamma\delta}$$

- The solution to the BSE is of the form: $\mathcal{T}(q)_{\alpha\beta,\gamma\delta} = \tau(q^2)\,\Omega_{\alpha\beta}\bar{\Omega}_{\gamma\delta}$

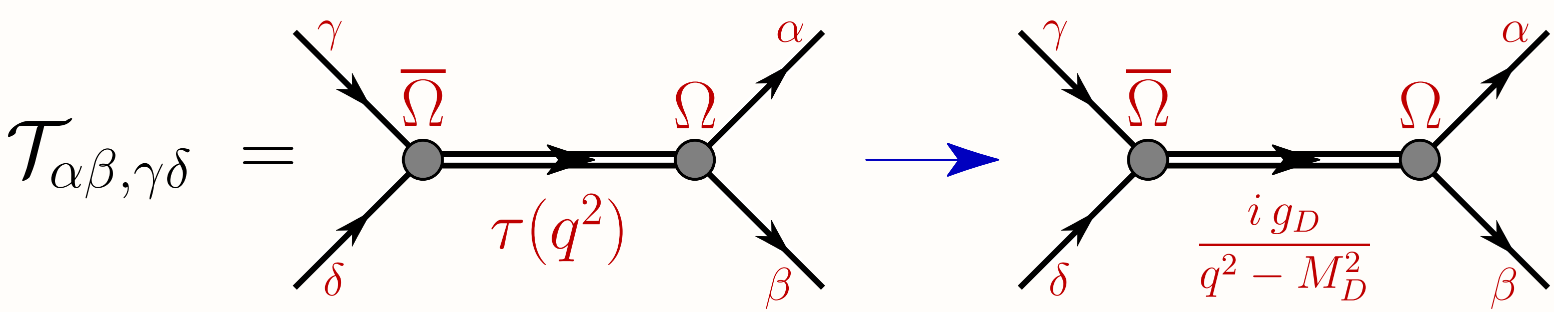
$$\tau_s(q^2) = \frac{4i\,G_s}{1+2\,G_s\,\Pi_s(q^2)} \qquad \tau_a^{\mu\nu}(q) = \frac{4\,i\,G_a}{1+2\,G_a\,\Pi_a(q^2)}\left[g^{\mu\nu} + 2\,G_a\,\Pi_a(q^2)\,\frac{q^\mu q^\nu}{q^2}\right]$$

# Diquark Propagators

- The reduced $t$-matrices are the diquark propagators

$$\tau_s(q^2) = \frac{4i\,G_s}{1+2\,G_s\,\Pi_s(q^2)} \qquad \tau_a^{\mu\nu}(q) = \frac{4\,i\,G_a}{1+2\,G_a\,\Pi_a(q^2)}\left[g^{\mu\nu} + 2\,G_a\,\Pi_a(q^2)\,\frac{q^\mu q^\nu}{q^2}\right]$$

- Near the pole they behave as elementary propagators




- The diquark masses are therefore given by

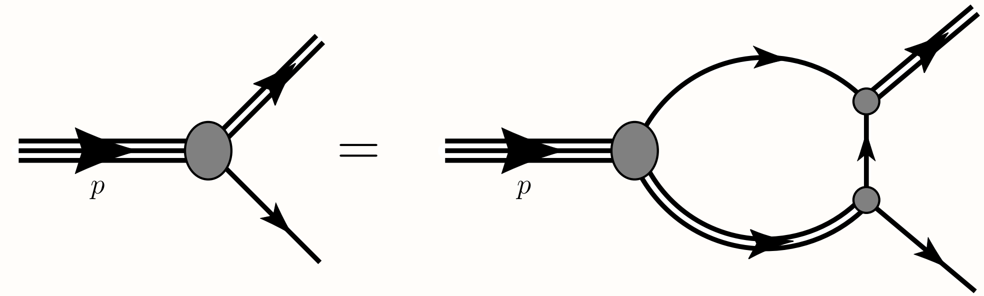
$$1 + 2\,G_s\,\Pi_s(q^2 = M_s^2) = 0 \qquad 1 + 2\,G_a\,\Pi_a(q^2 = M_a^2) = 0$$

- Expanding $\Pi(q^2)$ near the pole gives the quark-diquark coupling constant

$$g_D^2 = -\frac{2}{\frac{\partial}{\partial q^2}\,\Pi_D(q^2)}\bigg|_{q^2=M_D^2}$$

# The NJL Faddeev Equation

- To describe nucleon Faddeev equation kernel must be projected onto colour singlet, spin one-half, isospin one-half & positive parity




- Make the "static approximation" to quark exchange kernel: $S(p) \to -\frac{1}{M}$
- Homogeneous Faddeev amplitude with static approximation does not depend of relative momentum between the quark and diquark
- The Faddeev equation and vertex have the form

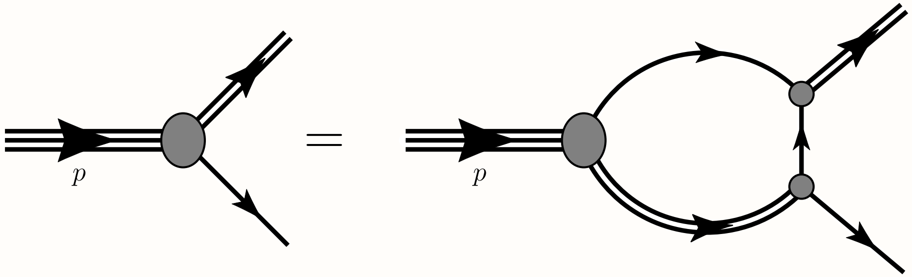
$$\Gamma_N(p,s) = K(p)\,\Gamma_N(p,s)$$

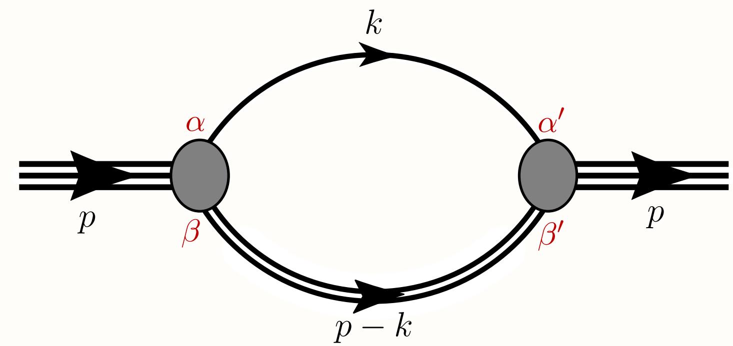
$$\Gamma_N(p,s) = \sqrt{-Z_N\frac{M_N}{p_0}}\left[\overset{\alpha_1}{\phantom{x}}\alpha_2\,\frac{p^\mu}{M_N}\,\gamma_5 + \alpha_3\,\gamma^\mu\gamma_5\right]u_N(p,s)$$

  - ✦ $K(p)$ is the Faddeev kernel

- Faddeev equation describes the continual recombination of the three quark in quark-diquark configurations

# The NJL Faddeev Equation (2)




- The kernel of this NJL Faddeev eq – $\Gamma_N(p,s) = K(p)\,\Gamma_N(p,s)$ – is

$$\begin{bmatrix}\Gamma_s \\ \Gamma_a^{\mu}\end{bmatrix} = \frac{3}{M}\begin{bmatrix}\Pi_{Ns} & \sqrt{3}\gamma_\alpha\gamma_5\,\Pi_{Na}^{\alpha\beta} \\ \sqrt{3}\gamma_5\gamma^{\mu}\,\Pi_{Ns} & -\gamma_\alpha\gamma^{\mu}\,\Pi_{Na}^{\alpha\beta}\end{bmatrix}\begin{bmatrix}\Gamma_s \\ \Gamma_{a,\beta}\end{bmatrix}$$

- The quark-diquark bubble diagrams are

$$\Pi_{Ns}(p) = \int\frac{d^4k}{(2\pi)^4}\,\tau_s(p-k)\,S(k)$$

$$\Pi_{Na}^{\mu\nu}(p) = \int\frac{d^4k}{(2\pi)^4}\,\tau_a^{\mu\nu}(p-k)\,S(k)$$

- Can now solve for the coefficients – $\alpha_1,\ \alpha_2, \alpha_3$
  - ✦ this then gives the NJL Faddeev amplitude

# *DSE Faddeev Equation*

- The DSE Faddeev equation has far more structure than in NJL
- For example the DSE Faddeev equation including scalar and axial-vector diquarks reads

$$\begin{bmatrix} \mathcal{S}(k,P) \\ \mathcal{A}_i^{\mu}(k,P) \end{bmatrix} u_N(p) = \int \frac{d^4\ell}{(2\,\pi)^4}\, \mathcal{M}_{ij}^{\mu\nu}(\ell;k,P) \begin{bmatrix} \mathcal{S}(\ell,P) \\ \mathcal{A}_\nu^{j}(\ell,P) \end{bmatrix} u_N(p)$$

  - ✦ importantly the vertex function depends on the relative momentum, $k$, between the quark and diquark
  - ✦ the Faddeev kernel is $\mathcal{M}_{ij}^{\mu\nu}(\ell;k,P)$
  - ✦ $\mathcal{S}(k,P)$ and $\mathcal{A}_i^{\mu}(k,P)$ describe the momentum space correlation between the quark and diquark in the nucleon

- This equation can be solved numerically on a large 2-$D$ grid in $k$ and $P$
- However standard practice to use a Chebyshev expansion for $\mathcal{S}(k,P)$ & $\mathcal{A}_i^{\mu}(k,P)$ and the solve for the coefficients of the expansion
  - ✦ a Chebyshev expansion is an expansion in Chebyshev polynomials

# Table of Contents