

Hadron Phenomenology and QCDs DSEs

Lecture 2: *The NJL model – from current to constituent quarks*

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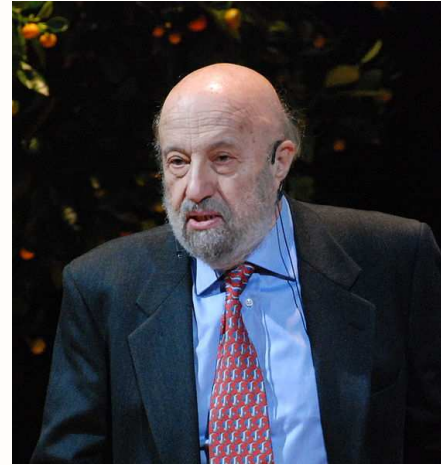
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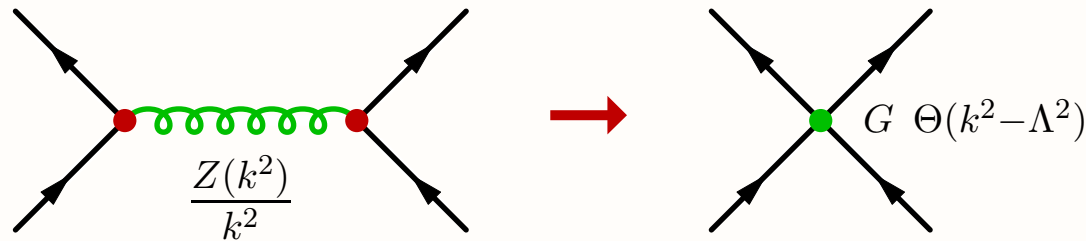
The Nambu–Jona-Lasinio Model



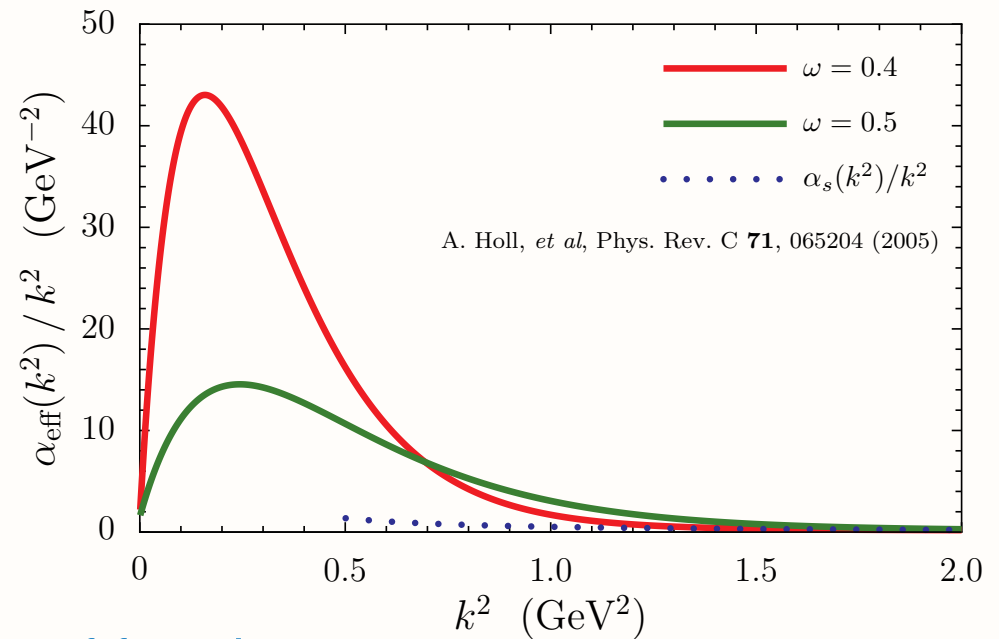
- The Nambu–Jona-Lasinio (NJL) Model was invented in 1961 by *Yoichiro Nambu* and *Giovanni Jona-Lasinio* while at The University of Chicago
 - ◆ was inspired by the BCS theory of superconductivity
 - ◆ was originally a theory of elementary nucleons
 - ◆ rediscovered in the 80s as an effective quark theory
- It is a relativistic quantum field theory, that is relatively easy to work with, and is very successful in the description of hadrons, nuclear matter, etc
- Nambu won half the 2008 Nobel prize in physics in part for the NJL model: *“for the discovery of the mechanism of spontaneous broken symmetry in subatomic physics”* [Nobel Committee]

NJL Model

- NJL model is interpreted as low energy chiral effective theory of QCD



- Can be motivated by infrared enhancement of quark-gluon interaction
e.g. DSEs and Lattice QCD

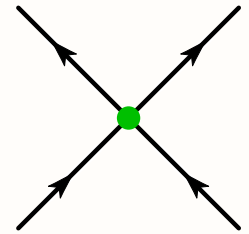


- Investigate the role of quark degrees of freedom
- NJL has same flavour symmetries as QCD
- NJL is non-renormalizable \implies cannot remove regularization parameter
- We will contrast NJL results with full DSE results

NJL Lagrangian

- In general the NJL Lagrangian has the form

$$\mathcal{L}_{NJL} = \mathcal{L}_0 + \mathcal{L}_I = \bar{\psi} (i \not{\partial} - m) \psi + \sum_{\alpha} G_{\alpha} (\bar{\psi} \Gamma_{\alpha} \psi)^2$$



- ◆ Γ_{α} represents a product of Dirac, colour and flavour matrices
- What about \mathcal{L}_I ? – effective theories should maintain symmetries of QCD
- In chiral limit QCD Lagrangian has symmetries

$$\mathcal{S}_{QCD} = SU(3)_c \otimes SU(N_f)_V \otimes SU(N_f)_A \otimes U(1)_V \otimes U(1)_A \otimes \mathcal{C} \otimes \mathcal{P} \otimes \mathcal{T}$$

- ◆ $SU(N_f)_A$ is broken dynamically – DCSB
- ◆ $U(1)_A$ is broken in the anomalous mode – $U(1)$ problem – massive η'
- NJL interaction Lagrangian must respect the symmetries

$$\mathcal{S}_{NJL} = SU(3)_c \otimes SU(N_f)_V \otimes SU(N_f)_A \otimes U(1)_V \otimes \mathcal{C} \otimes \mathcal{P} \otimes \mathcal{T}$$

- ◆ in NJL $SU(3)_c$ will be considered a global gauge symmetry
- ◆ $U(1)_A$ is often broken explicitly $\implies m_{\eta'} \neq 0$

NJL Lagrangian (2)

$$\mathcal{S}_{NJL} = SU(3)_c \otimes SU(N_f)_V \otimes SU(N_f)_A \otimes U(1)_V \otimes \mathcal{C} \otimes \mathcal{P} \otimes \mathcal{T}$$

- The NJL Lagrangian should be symmetric under the transformations

$$\begin{aligned} SU(N_f)_V : \quad \psi &\longrightarrow e^{-i \mathbf{t} \cdot \boldsymbol{\theta}_V} \psi & \bar{\psi} &\longrightarrow \bar{\psi} e^{i \mathbf{t} \cdot \boldsymbol{\theta}_V} \\ SU(N_f)_A : \quad \psi &\longrightarrow e^{-i \gamma_5 \mathbf{t} \cdot \boldsymbol{\theta}_A} \psi & \bar{\psi} &\longrightarrow \bar{\psi} e^{-i \gamma_5 \mathbf{t} \cdot \boldsymbol{\theta}_A} \\ U(1)_V : \quad \psi &\longrightarrow e^{-i \theta} \psi & \bar{\psi} &\longrightarrow \bar{\psi} e^{i \theta} \\ U(1)_A : \quad \psi &\longrightarrow e^{-i \gamma_5 \theta} \psi & \bar{\psi} &\longrightarrow \bar{\psi} e^{-i \gamma_5 \theta} \end{aligned}$$

- Nambu and Jona-Lasinio choose the Lagrangian

$$\mathcal{L} = \bar{\psi} (i \not{\partial} - m) \psi + G_\pi \left[(\bar{\psi} \psi)^2 - (\bar{\psi} \gamma_5 \boldsymbol{\tau} \psi)^2 \right]$$



- Can choose any combination of these 4-fermion interactions

$$\begin{aligned} &(\bar{\psi} \psi)^2, \quad (\bar{\psi} \gamma_5 \psi)^2, \quad (\bar{\psi} \gamma^\mu \psi)^2, \quad (\bar{\psi} \gamma^\mu \gamma_5 \psi)^2, \quad (\bar{\psi} i \sigma^{\mu\nu} \psi)^2, \\ &(\bar{\psi} \mathbf{t} \psi)^2, \quad (\bar{\psi} \gamma_5 \mathbf{t} \psi)^2, \quad (\bar{\psi} \gamma^\mu \mathbf{t} \psi)^2, \quad (\bar{\psi} \gamma^\mu \gamma_5 \mathbf{t} \psi)^2, \quad (\bar{\psi} i \sigma^{\mu\nu} \mathbf{t} \psi)^2. \end{aligned}$$

NJL Lagrangian (3)

- The most general $N_f = 2$ NJL Lagrangian that respects the symmetries is

$$\mathcal{L} = \bar{\psi} (i\not{\partial} - m) \psi + G_\pi \left[(\bar{\psi}\psi)^2 - (\bar{\psi} \gamma_5 \boldsymbol{\tau} \psi)^2 \right] + G_\omega (\bar{\psi} \gamma^\mu \psi)^2 + G_\rho \left[(\bar{\psi} \gamma^\mu \boldsymbol{\tau} \psi)^2 + (\bar{\psi} \gamma^\mu \gamma_5 \boldsymbol{\tau} \psi)^2 \right] \\ + G_h (\bar{\psi} \gamma^\mu \gamma_5 \psi)^2 + G_\eta \left[(\bar{\psi} \gamma_5 \psi)^2 - (\bar{\psi} \boldsymbol{\tau} \psi)^2 \right] + G_T \left[(\bar{\psi} i\sigma^{\mu\nu} \psi)^2 - (\bar{\psi} i\sigma^{\mu\nu} \boldsymbol{\tau} \psi)^2 \right]$$

- ◆ \mathcal{L}_I is $U(1)_A$ invariant if: $G_\pi = -G_\eta$ & $G_T = 0$

$\bar{\psi}\psi$	\longleftrightarrow	σ	\longleftrightarrow	$(J^P, T) = (0^+, 0)$
$\bar{\psi} \gamma_5 \boldsymbol{\tau} \psi$	\longleftrightarrow	π	\longleftrightarrow	$(J^P, T) = (0^-, 1)$
$\bar{\psi} \gamma^\mu \psi$	\longleftrightarrow	ω	\longleftrightarrow	$(J^P, T) = (1^-, 0)$
$\bar{\psi} \gamma^\mu \boldsymbol{\tau} \psi$	\longleftrightarrow	ρ	\longleftrightarrow	$(J^P, T) = (1^-, 1)$
$\bar{\psi} \gamma^\mu \gamma_5 \psi$	\longleftrightarrow	f_1, h_1	\longleftrightarrow	$(J^P, T) = (1^+, 0)$
$\bar{\psi} \gamma^\mu \gamma_5 \boldsymbol{\tau} \psi$	\longleftrightarrow	a_1	\longleftrightarrow	$(J^P, T) = (1^+, 1)$
$\bar{\psi} \boldsymbol{\tau} \psi$	\longleftrightarrow	a_0	\longleftrightarrow	$(J^P, T) = (0^+, 1)$
$\bar{\psi} \gamma_5 \psi$	\longleftrightarrow	η, η'	\longleftrightarrow	$(J^P, T) = (0^-, 0)$

NJL Lagrangian (4)

- The most general $N_f = 2$ NJL Lagrangian that respects the symmetries is

$$\mathcal{L}_I = \frac{1}{2} G_\pi \left[(\bar{\psi}\psi)^2 - (\bar{\psi} \gamma_5 \boldsymbol{\tau} \psi)^2 \right] - \frac{1}{2} G_\omega (\bar{\psi} \gamma^\mu \psi)^2 - \frac{1}{2} G_\rho \left[(\bar{\psi} \gamma^\mu \boldsymbol{\tau} \psi)^2 - (\bar{\psi} \gamma^\mu \gamma_5 \boldsymbol{\tau} \psi)^2 \right] \\ + \frac{1}{2} G_f (\bar{\psi} \gamma^\mu \gamma_5 \psi)^2 - \frac{1}{2} G_\eta \left[(\bar{\psi} \gamma_5 \psi)^2 - (\bar{\psi} \boldsymbol{\tau} \psi)^2 \right] - \frac{1}{2} G_T \left[(\bar{\psi} i\sigma^{\mu\nu} \psi)^2 - (\bar{\psi} i\sigma^{\mu\nu} \boldsymbol{\tau} \psi)^2 \right]$$

- ◆ \mathcal{L}_I is $U(1)_A$ invariant if: $G_\pi = -G_\eta$ & $G_T = 0$

- The most general $N_f = 3$ NJL Lagrangian that respects the symmetries is

$$\mathcal{L}_I = G_\pi \left[\frac{1}{6} (\bar{\psi}\psi)^2 + (\bar{\psi} \mathbf{t} \psi)^2 - \frac{1}{6} (\bar{\psi} \gamma_5 \psi)^2 - (\bar{\psi} \gamma_5 \mathbf{t} \psi)^2 \right] \\ - \frac{1}{2} G_\rho \left[(\bar{\psi} \gamma^\mu \mathbf{t} \psi)^2 + (\bar{\psi} \gamma^\mu \gamma_5 \mathbf{t} \psi)^2 \right] - \frac{1}{2} G_\omega (\bar{\psi} \gamma^\mu \psi)^2 - \frac{1}{2} G_f (\bar{\psi} \gamma^\mu \gamma_5 \psi)^2$$

- Enlarging the $SU(N_f)_V \otimes SU(N_f)_A$ chiral group from $N_f = 2$ to $N_f = 3$ reduces the number of coupling from six to four

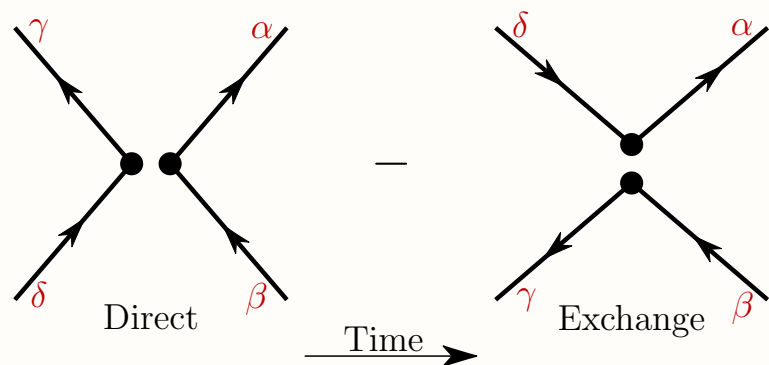
- The $N_f = 3$ Lagrangian is automatically $U(1)_A$ invariant

- ◆ $U(1)_A$ is then often broken by the 't Hooft term – a 6-quark interaction

$$\mathcal{L}_I^{(6)} = K \left[\det (\bar{\psi}(1 + \gamma_5)\psi) + \det (\bar{\psi}(1 - \gamma_5)\psi) \right]$$

NJL Interaction Kernel

- Using Wick's theorem and the NJL Lagrangian there are 2 diagrams for the interaction between a quark and an anti-quark



$$2i G \left[\Omega_{\alpha\beta}^i \bar{\Omega}_{\gamma\delta}^i - \Omega_{\alpha\delta}^i \bar{\Omega}_{\gamma\beta}^i \right]$$

- Using Fierz transformations can express each *exchange term* as a sum of *direct terms*
- The $SU(2)$ NJL interaction kernel then takes the form

$$K_{\alpha\beta,\gamma\delta} = 2i G_\pi \left[(\mathbb{1})_{\alpha\beta} (\mathbb{1})_{\gamma\delta} - (\gamma_5 \boldsymbol{\tau})_{\alpha\beta} (\gamma_5 \boldsymbol{\tau})_{\gamma\delta} \right] - 2i G_\omega (\gamma_\mu)_{\alpha\beta} (\gamma^\mu)_{\gamma\delta} \\ - 2i G_\rho \left[(\gamma_\mu \boldsymbol{\tau})_{\alpha\beta} (\gamma^\mu \boldsymbol{\tau})_{\gamma\delta} + (\gamma_\mu \gamma_5 \boldsymbol{\tau})_{\alpha\beta} (\gamma^\mu \gamma_5 \boldsymbol{\tau})_{\gamma\delta} \right] + \dots$$

- This kernel enters the NJL gap and meson Bethe-Salpeter equations

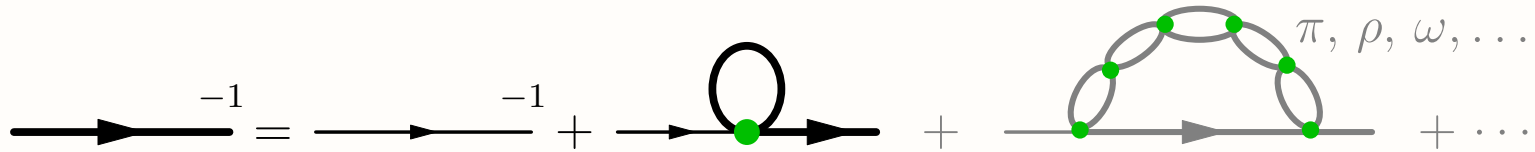
Regularization Schemes

- The NJL model is non-renormalizable \implies cannot remove regularization
 - ◆ regularization parameter(s) play a dynamical role
- Popular choices are:
 - ◆ 3-momentum cutoff: $\vec{p}^2 < \Lambda^2$
 - ◆ 4-momentum cutoff $p_E^2 < \Lambda^2$
 - ◆ Pauli-Villars, etc
- We will use the proper-time regularization scheme

$$\frac{1}{X^n} = \frac{1}{(n-1)!} \int_0^\infty d\tau \tau^{n-1} e^{-\tau X} \rightarrow \frac{1}{(n-1)!} \int_{1/\Lambda_{UV}^2}^{1/\Lambda_{IR}^2} d\tau \tau^{n-1} e^{-\tau X}$$

- ◆ only Λ_{UV} is need to render the theory finite
- ◆ however, as we shall see, Λ_{IR} plays a very important role; it prevents quarks going on their mass shell and hence **simulates confinement**

NJL Gap Equation



- The NJL gap equation has the form

$$S^{-1}(k) = S_0^{-1}(k) - \Sigma(k) = [\not{k} - m] - \sum_j \int \frac{d^4 \ell}{(2\pi)^4} \text{Tr} [S(\ell) \bar{\Omega}^j] \Omega^j$$

- The only piece of the interaction kernel that contributes is:

$$K_{\alpha\beta, \gamma\delta}^\sigma = 2i G_\pi (\mathbb{1})_{\gamma\delta} (\mathbb{1})_{\alpha\beta}$$

- Solving this equation give a quark propagator of the form

$$S^{-1}(k) = \not{k} - M + i\varepsilon$$

- The constituent quark mass satisfies the equation

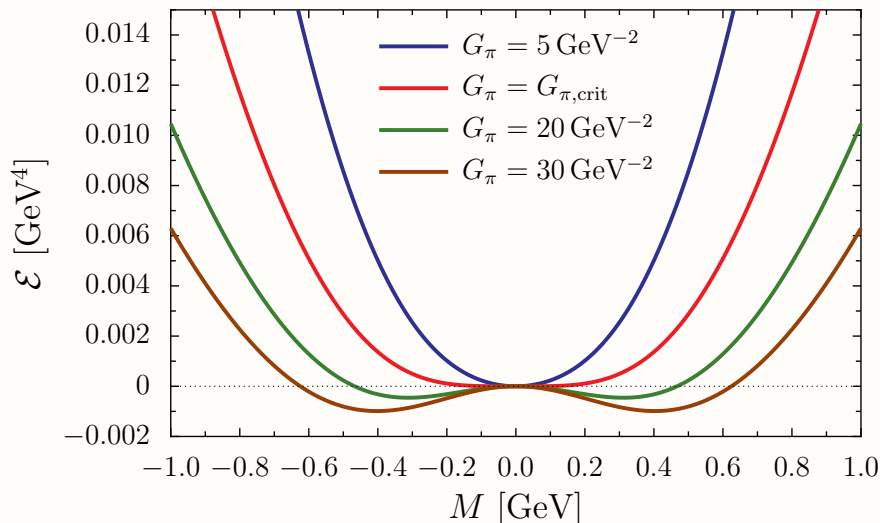
$$M = m + 48i G_\pi M \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{\ell^2 - M^2 + i\varepsilon} = m + M \frac{3 G_\pi}{\pi^2} \int d\tau \frac{e^{-\tau M^2}}{\tau^2}$$

NJL Gap Equation (2)

$$M = m + M \frac{3 G_\pi}{\pi^2} \int d\tau \frac{e^{-\tau M^2}}{\tau^2}$$

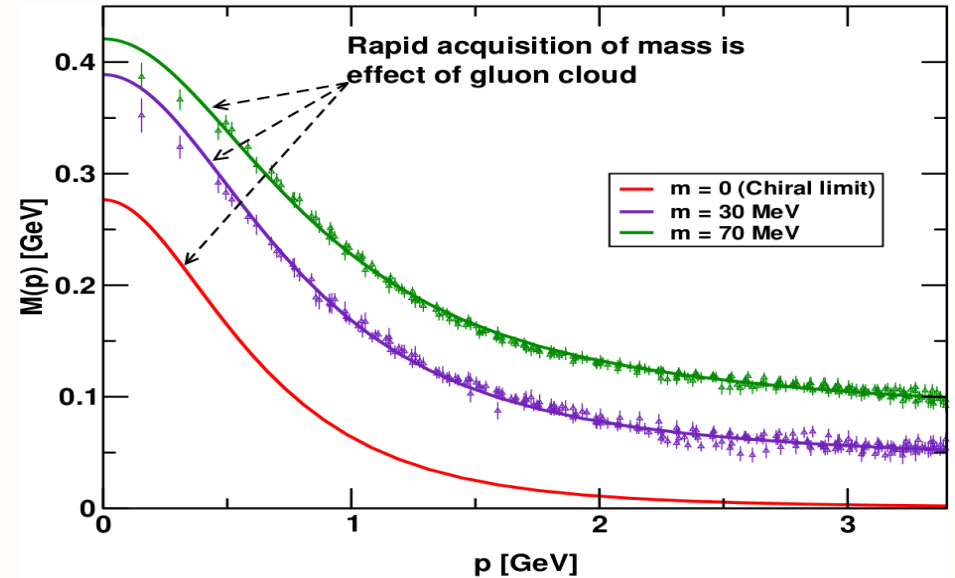
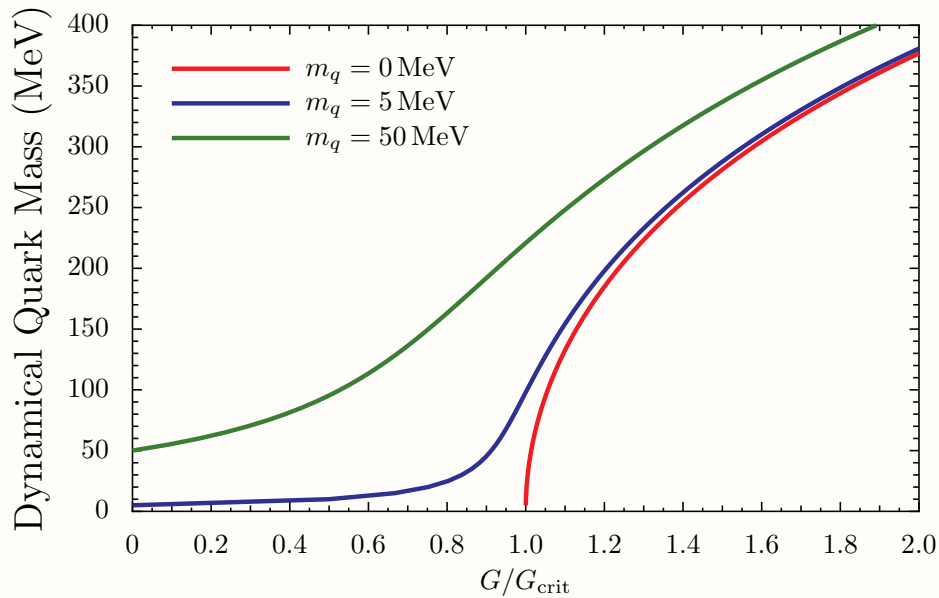
- For the case $m = 0$ the gap equation has two solutions:
 - ◆ trivial solution: $M = 0$ & non-trivial solution: $M \neq 0$
- Which solution does nature choose, that is, which solution minimizes the energy. Compare vacuum energy density, \mathcal{E} , for each case

$$\mathcal{E}(M) - \mathcal{E}(M = 0) = -\frac{3}{4\pi^2} \int d\tau \frac{1}{\tau^3} \left(e^{-\tau M^2} - 1 \right) + \frac{M^2}{4 G_\pi}$$



- For $G_\pi > G_{\pi,\text{crit}}$ the lowest energy solution has $M \neq 0$
- Therefore for $G_\pi > G_{\pi,\text{crit}}$ NJL has DCSB
- DCSB \iff generates mass from nothing

NJL & DSE gap equations



- NJL constituent mass is given by: $M = m - 2 G_{\pi} \langle \bar{\psi} \psi \rangle$

- Chiral condensate is defined by

$$\langle \bar{\psi} \psi \rangle \equiv \lim_{x \rightarrow y} \text{Tr} [-i S(x - y)] = - \int \frac{d^4 k}{(2\pi)^4} \text{Tr} [i S(k)]$$

- Mass is generated via interaction with vacuum

- Dynamically generated quark masses $\iff \langle \bar{\psi} \psi \rangle \neq 0$

- Difference in mass functions should have observable consequences!

Confinement in NJL model

- In general the NJL model is not confining; quark propagator is simply

$$S(k) = \frac{1}{\not{k} - M + i\varepsilon} = \frac{\not{k} + M}{k^2 - M^2 + i\varepsilon}$$

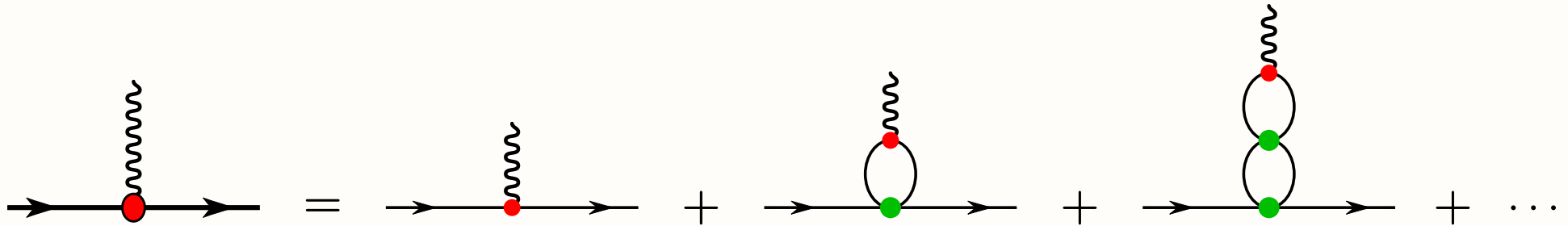
- ◆ quark propagator has a pole \implies quarks are part of physical spectrum
- However the proper-time scheme is unique

$$S(k) = \int_0^\infty d\tau (\not{k} + M) e^{-\tau(k^2 - M^2)} \rightarrow \underbrace{\frac{[e^{-\Lambda_{UV}(k^2 - M^2)} - e^{-\Lambda_{IR}(k^2 - M^2)}]}{k^2 - M^2}}_{\equiv Z(k^2)} [\not{k} + M]$$

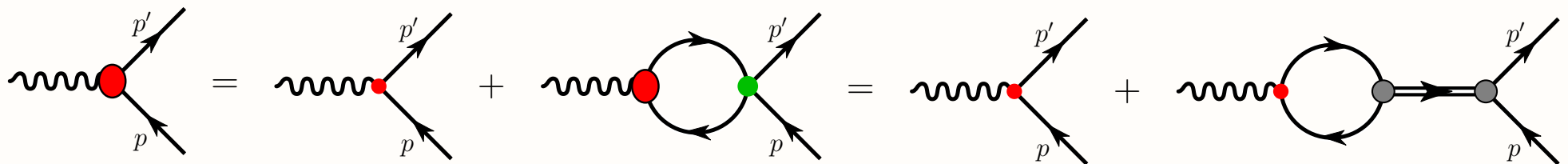
- quark propagator does not have a pole: $Z(k^2) \stackrel{k^2 \rightarrow M^2}{=} \Lambda_{IR} - \Lambda_{UV} \neq \infty$
- Are confinement and DCSB related?
 - ◆ NJL model is proof that DCSB can exist without confinement
 - ◆ however there is no example of model with confinement and no DCSB!

From Current to Constituent Quarks

- Both the DSE and NJL gap equations take current quarks and dress them non-perturbatively so that they become constituent quarks
- **Constituent quarks are extended non-trivial quasi-particles**
- Consider an arbitrary current interacting with the current quarks



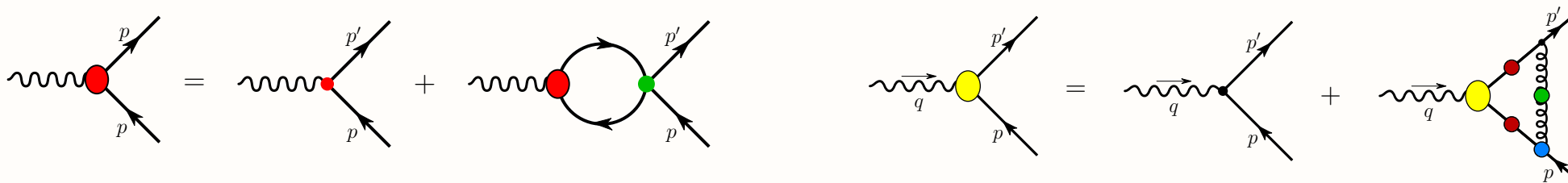
- This series can be represented by an integral equation



◆ *This is the inhomogeneous Bethe-Salpeter equation (BSE)*

Constituent Quark EM Form Factors

- The quark-photon vertex is given by the Bethe-Salpeter equation – where the driving term is an external vector current: $\gamma^\mu \left(\frac{1}{6} + \frac{\tau_3}{2} \right)$



- Lorentz covariance implies that the quark–photon vertex has the structure

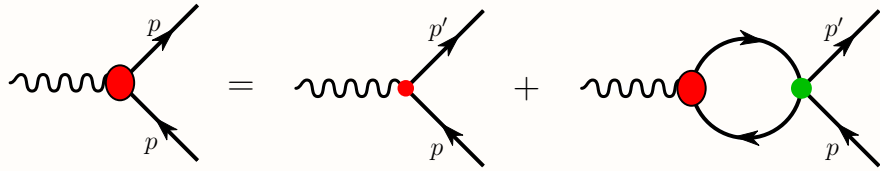
$$\Gamma_{\gamma qq}^\mu(p', p) = \sum_{i=1}^{12} \lambda_i^\mu f_i(p'^2, p^2, q^2) = \Gamma_L^\mu(p', p) + \Gamma_T^\mu(p', p)$$

- In QCD the properties of the quark–photon vertex are governed by the quark propagator and the quark–gluon vertex
- Ward-Takahashi identity constrains Γ_L^μ piece of quark–photon vertex

$$q_\mu \Gamma_{\gamma qq}^\mu = q_\mu \Gamma_L^\mu = \hat{Q} [S^{-1}(p') - S^{-1}(p)], \quad q_\mu \Gamma_T^\mu = 0$$

- This BSE is difficult to solve in DSE framework, however in the NJL model it is straight forward

NJL Constituent Quark Form Factors



$$K_{\alpha\beta,\gamma\delta} = -2i G_\omega (\gamma_\mu)_{\alpha\beta} (\gamma^\mu)_{\gamma\delta} - 2i G_\rho (\gamma_\mu \boldsymbol{\tau})_{\alpha\beta} (\gamma^\mu \boldsymbol{\tau})_{\gamma\delta}$$

- In general the quark–photon vertex has form

$$\Gamma_{\gamma qq}^\mu(p', p) = \frac{1}{6} \Lambda_\omega^\mu(p', p) + \frac{\tau_3}{2} \Lambda_\rho^\mu(p', p).$$

- Recall Ward–Takahashi identity $[S^{-1}(p) = \not{p} - M + i\varepsilon]$

$$q_\mu \Gamma_{\gamma qq}^\mu(p', p) = \left(\frac{1}{6} + \frac{\tau_3}{2} \right) [S^{-1}(p') - S^{-1}(p)] \xrightarrow{NJL} \left(\frac{1}{6} + \frac{\tau_3}{2} \right) \not{q}$$

- NJL the vertex must be of form $\Lambda_{\omega,\rho}^\mu = \gamma^\mu + \text{transverse terms}$
- Solving the NJL inhomogeneous BSE for the quark–photon vertex gives

$$\Lambda_\omega^\mu(p', p) = \gamma^\mu + \left(\gamma^\mu - \frac{q^\mu \not{q}}{q^2} \right) \hat{F}_{1\omega}(q^2), \quad \Lambda_\rho^\mu(p', p) = \gamma^\mu + \left(\gamma^\mu - \frac{q^\mu \not{q}}{q^2} \right) \hat{F}_{1\rho}(q^2)$$

NJL Results

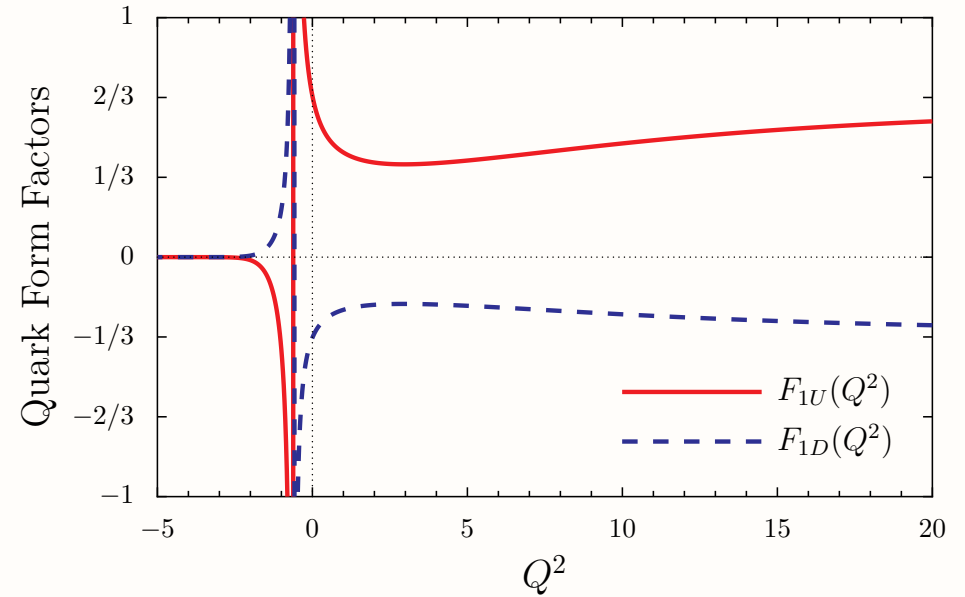
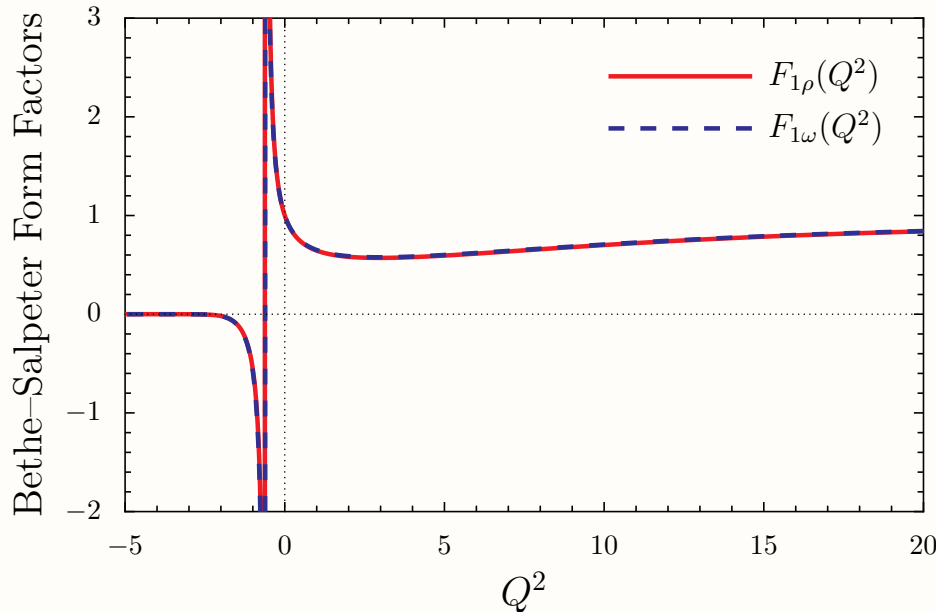
- Putting the quark-photon vertex on-shell gives

$$\langle J^\mu \rangle = \bar{u}(p') \Gamma_{\gamma qq}^\mu u(p) = \gamma^\mu \frac{1}{6} [1 + \hat{F}_{1\omega}] + \gamma^\mu \frac{\tau_3}{2} [1 + \hat{F}_{1\rho}] \equiv \gamma^\mu \left[\frac{1}{6} F_{1\omega} + \frac{\tau_3}{2} F_{1\rho} \right]$$

- The up and down constituent quark form factors are given by $[Q^2 = -q^2]$

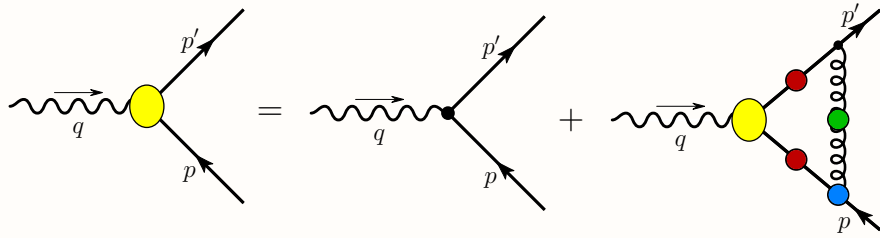
$$F_{1U}(Q^2) = \frac{1}{6} F_{1\omega}(Q^2) + \frac{1}{2} F_{1\rho}(Q^2) \quad \& \quad F_{1D}(Q^2) = \frac{1}{6} F_{1\omega}(Q^2) - \frac{1}{2} F_{1\rho}(Q^2)$$

- Timelike poles at: $F_{1\omega}(Q^2 = -m_\omega^2)$ & $F_{1\rho}(Q^2 = -m_\rho^2)$



$$F_{1\omega}(Q^2) = \frac{1}{1+2 G_\omega \Pi_{VV}(Q^2)} \quad \& \quad F_{1\rho}(Q^2) = \frac{1}{1+2 G_\rho \Pi_{VV}(Q^2)}$$

DSE Quark Form Factors



$$q_\mu \Gamma_{\gamma qq}^\mu(p', p) = \hat{Q} [S^{-1}(p') - S^{-1}(p)]$$

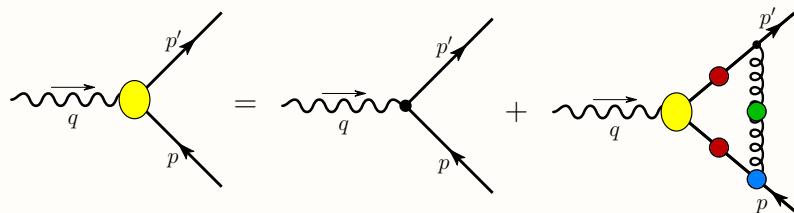
- The longitudinal piece of the quark-photon vertex, $\Gamma_{\gamma qq}^\mu = \Gamma_L^\mu + \Gamma_T^\mu$, is completely determined by the quark propagator
- This result is encapsulated in the Ball-Chiu vertex

$$\Gamma_{BC}^\mu = \frac{A(p'^2) + A(p^2)}{2} \gamma^\mu - \frac{A(p'^2) - A(p^2)}{p'^2 - p^2} i(p' + p)^\mu + \frac{1}{2} \frac{A(p'^2) - A(p^2)}{p'^2 - p^2} (\not{p}' + \not{p})(p' + p)^\mu$$

- Recall: $S^{-1}(p) = i\not{p} A(p^2) + B(p^2)$ – it is then straight forward to show Γ_{BC}^μ satisfies the WTI
- The nature of the quark-photon vertex is largely controlled by the structure of the quark-gluon vertex
 - ◆ different quark-gluon vertices can give very similar quark-propagators
 - ◆ therefore transverse piece of $\Gamma_{\gamma qq}^\mu$ sensitive to the quark-gluon vertex
- Recall rainbow ladder: $\Gamma_{gqq}^{a,\mu} = \frac{\lambda^a}{2} \gamma^\mu$

DSE Quark Anomalous Magnetic Moment

- Include $\sigma^{\mu\nu} q_\nu \tau_5(p', p)$ [anomalous chromomagnetic] term in quark–gluon vertex: $\Gamma_{gqq}^{a,\mu}(p', p) = \frac{\lambda^a}{2} [\gamma^\mu + \sigma^{\mu\nu} q_\nu \tau_5(p', p)]$
 - ◆ beyond rainbow ladder – has been absent from previous calculations
- Generates anomalous electromagnetic term in quark–photon vertex
- Confined quarks \implies no mass shell – anomalous mm ill defined
 - ◆ however associate with $i\sigma^{\mu\nu} q_\nu$ piece of quark–photon vertex



- L. Chang, Y. -X. Liu, C. D. Roberts, Phys. Rev. Lett. **106**, 072001 (2011).
- Investigate effect on nucleon form factors

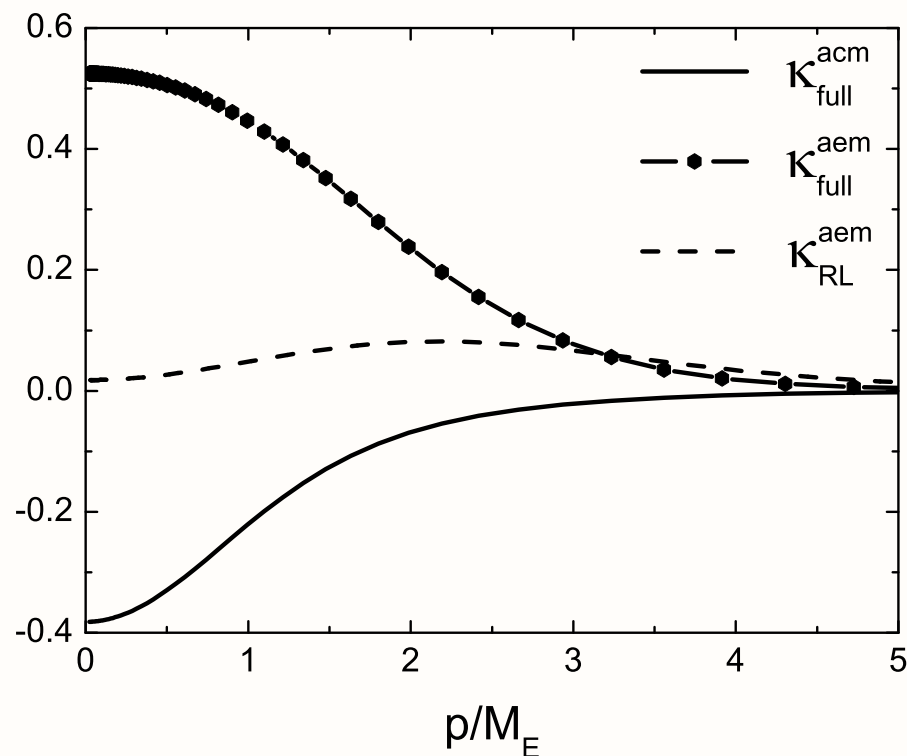


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