Using the Covariant Spectator Theory® (CST) to extract N* properties at high $Q^2$

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★ Part I -- What is our goal?
  • Equivalence between quark-gluon and hadronic degrees of freedom
  • The goal is to determine the hadronic lagrangian

★ Part II -- Using the CST to model hadronic vertices at high $Q^2$
  • Compute BARE $\gamma^* + N \to N^*$ vertices from the N and N* wave functions and quark form factors

★ Part III -- Connection to DIS

Thanks to:
Yohanes Surya
Gilberto Ramalho
Teresa Pena
PART I: What is our goal?
What is our goal?

1. Fundamental theoretical assumption (proved?):
   - If [quarks and gluons] ⇔ [baryons and mesons (hadrons)]
     are COMPLETELY EQUIVALENT descriptions of the physics, then
   - What are the REQUIRED hadronic fields in the Lagrangian (in order that equivalence will work)?
     - N, Δ, ●, ●, ●, ●, ● ["elementary" baryons]
     - π, ρ, η, σ(?), ●, ● ["elementary" mesons]

2. Using an accepted (dynamical and regularization) SCHEME compute
   \[ γ^* + N \rightarrow π + N; \]
   \[ γ^* + N \rightarrow ρ + N; \]
   \[ γ^* + N \rightarrow π + π + N; \]
   \[ γ^* + N \rightarrow ●, ●, ●. \]

2. The goal is to use the accepted SCHEME to find the bare baryon poles corresponding to the “elementary” baryons that appear in the hadronic Lagrangian: -- and to DETERMINE THE HADRONIC LAGRANGIAN
Lessons from the history of the $\Delta$

★ Chew-Low theory (1955)
  • NO elementary $\Delta$ pole
  • $\Delta$ resonance generated by N exchange diagram (u channel pole)

★ Bootstrap: resonance feedback (1960's)

★ Discovery of quarks and recognition that the $\Delta$ is "elementary"
  • (i.e. it is REQUIRED for equivalence between hadronic and quark degrees of freedom)

★ The $\Delta$ - N mass difference enters into many estimates. Is it the “bare” mass difference or the “dressed” mass difference?

★ Conclusion: The existence of a bare delta pole is crucial to our understanding
## Bare poles from various models (incomplete survey!)

<table>
<thead>
<tr>
<th>Reference</th>
<th>$\Delta$ (1232) bare mass</th>
<th>$D_{13}$ (1520) bare mass</th>
<th>$P_{11}$ (1440) bare mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sato &amp; Lee (1996)</td>
<td>1299.07 (model L)</td>
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<td>1318.52 (model H)</td>
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<td>Suzuki, et.al. (2010)</td>
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<tr>
<td>Döring, et al. (2011)</td>
<td>1535 (2011)</td>
<td></td>
<td>NO POLE</td>
</tr>
</tbody>
</table>
WARNING:

Our physical insight and understanding depends on our computational scheme

EXAMPLE: the “angular momentum theorem”
Does $F_2 > 0$ require $\ell > 0$? [Angular momentum theorem]

Answer to this question depends on the formalism. There are two points of view:

(a) 

(b)
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CST view: All interactions involving gluon exchange between the $q\bar{q}$ pair coupled to the photon are included in quark form factors; these produce the quark anomalous moments.
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**Light-front view:** Nucleon wave function is a sum over Fock components; quark "structure" comes from higher Fock components of the hadronic wave function.
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**CST view:** All interactions involving gluon exchange between the $q\bar{q}$ pair coupled to the photon are included in quark form factors; these produce the quark anomalous moments.

**Light-front view:** Nucleon wave function is a sum over Fock components; quark “structure” comes from higher Fock components of the hadronic wave function.

The light-front view requires $\ell > 0$ components just to give $\kappa_\pm \neq 0$.
PART II: Using the Covariant Spectator Theory© (CST) to model the high $Q^2$ behavior of

$\gamma^* + N \rightarrow N^*$ hadronic vertices
The nucleon consists of 3 constituent quarks (CQ) with a size, mass, and form factor given by the dressing of the quark in the sea of gluons and $qq$ pairs. The sea quarks can be neglected.

Using the Covariant Spectator© theory, the nucleon and $\Delta$ is described by a 3-CQ vertex function with two of the CQ on shell

$$\Psi_\alpha = \left( \frac{1}{m - \not{P} - i\epsilon} \right)_{\alpha\beta} \Gamma_\beta (P, p_s)$$

Confinement insures that this vertex function is zero when all three quarks are on shell (i.e. there is no 3q scattering)

How should the wave function be modeled? Ockham's razor: Start with the simplest case -- pure $S$-state with same spin-isospin structure as the nonrelativistic wave function. See if it works!
Structure of the model baryon wave functions

- For non-strange systems (u and d quarks only) fermion antisymmetry comes from the color factor $\mathcal{E}_{\alpha\beta\gamma}$. The rest of the wf must be symmetric.

- Construct nonrelativistic wave functions first; then generalize by replacing (for example)
  
  \[ \mathbf{k} \rightarrow k^\mu - \frac{(k \cdot P) P^\mu}{P^2} \]

  \[ \delta^{ij} \rightarrow -g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2} \]

- Starting from a simple $S$ state spatial wave function, spin-isospin structure must be symmetric. Spin-isospin 1/2 requires superposition of mixed symmetry states
  
  \[ \left| p \uparrow \right> = \frac{1}{\sqrt{2}} \left\{ \Phi^0_F \Phi^0_S + \Phi^1_F \Phi^1_S \right\} \]

  Spin-isospin 3/2 requires pure symmetric states
  
  \[ \left| \Delta \uparrow \right> = \Phi^1_F \Phi^1_S \]

- When angular momentum components are added, F no longer necessarily equal to S.
Relativistic impulse approximation for the form factors

In the spectator theory, the photon couples to the off-shell quark, and because of the symmetry, the coupling to all three quarks is 3 times the coupling to one

\[ P_{\pm} = P \pm \frac{1}{2} q \quad Q^2 = -q^2 \]
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approximate the two spectator quarks by a single diquark with a fixed mass

\[ J^\mu = 3 \sum_\lambda \int \frac{d^3 k}{(2\pi)^3 2E_s(k)} \bar{\Psi}_f(P_+,k) j^\mu_l(q) \Psi_i(P_-,k) \]

Integrate over the (on-shell) spectator three momentum

Quark current with form factors
Historical overview of our work

- Pure S-wave nucleon wave function shown to fit the nucleon form factors
  - Assumes constituent quarks with form factors
  - Does not violate of the “angular momentum theorem” because quarks have structure (recall previous slide)

- This wave function used to explain (or predict) transition form factors for $\gamma^* + N \rightarrow N^*$ where $N^* = \Delta(1236), P_{11}(1440), S_{11}(1535)$, and $\Delta(1600)$

- Calculations have been extended to the strange sector.

- Recently, we extracted a new N wave function (without any pion cloud contributions) directly from DIS. Best model gives 35% D-state!!

- Next generation of calculations:
  - Use N wave function extracted from DIS
  - Calculate pion cloud using constraints obtained from octet magnetic moments
  - Revise the $\gamma^* + N \rightarrow N^*$ calculations
Nucleon form factors with S-wave nucleons (2008)*

- N wave function is S-wave only: 2 parameters
- Quark form factors:
  \[ f_{1+} = \lambda + \frac{1 - \lambda}{1 + Q_0^2/m_r^2} + \frac{c_+ Q_0^2/M_h^2}{(1 + Q_0^2/M_h^2)^2} \]
  2 parameters \( \lambda \) fixed by DIS \( c_+ \) fit
  \[ f_{2+} = \kappa_+ \left( \frac{d_+}{1 + Q_0^2/m_r^2} + \frac{(1 - d_+)}{1 + Q_0^2/M_h^2} \right) \]
  1 parameter \( \kappa_+ \) fixed by moments \( d_+ = d_- \) fit
- Blue lines: NO pion cloud
- Best phenomenology (5 parameters)
- Red lines: with a pion cloud
- Lesson: S-waves can explain data

*FG, Ramalho, Pena, PRC 77, 015202 (2008)
Results: $\gamma^* + N \rightarrow \Delta$ transition with PURE S-wave states

- Three form factors, but ONLY $G^*_M \neq 0$ if BOTH the $N$ and $\Delta$ wave functions are pure $S$-wave. $\Delta$ wave function has two new range parameters.
- The value $G^*_M(0)$ cannot equal the correct value unless a separate pion cloud term is added, because of the Schwartz inequality
  \[
  G^*_M(0) = \frac{8}{3\sqrt{3}} \left( \frac{m}{M+m} \right) j \int \psi_\Lambda \psi_N \\
  = 2.07 \int \psi_\Lambda \psi_N, \quad \text{and} \\
  \int \psi_\Lambda \psi_N \leq \sqrt{\int |\psi_N|^2} \sqrt{\int |\psi_\Lambda|^2} \leq 1
  \]
- Fit done with an empirical pion cloud term of the form
  \[
  \frac{G^*_M}{3G_D} = \lambda_\pi \left( \frac{\Lambda_{\pi}^2}{\Lambda_{\pi}^2 + Q^2} \right)^2
  \]
- Lessons:
  - quark core dominates large $Q^2$ region
  - Bare contributions agree with EBAC analysis

Results: $\gamma^* + N \to \Delta$ transition with D-waves

- Determine core D-wave admixtures by fitting $G_c^*$ and $G_E^*$ (0 for pure S-waves) to lattice data.
- Then add pion cloud to fit experimental data.
- Lesson: lattice data can be used to fix quark core when both the core and the pion cloud are important.

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*Ramalho and Pena, PRD 80, 013008 (2009)
Results: $\gamma^* + N \rightarrow P_{11} (1440)^*$

- One extra parameter in $N^*$ wave function fixed by orthogonality condition ($N^*$ is a radial excitation of $N$)
- Quark core transition amplitude fits high $Q^2$ data
- Pion cloud is estimated to be the difference between the MAID fit and the quark core (with error bands taken from the error bars in the data)

*Ramalho & Tsushima, PRD 81, 074020 (2010)
Results: $\gamma^* + N \rightarrow P_{11} (1440)$

- Results for the helicity amplitudes
- Zero crossing for $P_{11}$ must come from the pion cloud

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*Ramalho & Tsushima, PRD 81, 074020 (2010)*
Results: $\gamma^* + N \rightarrow \Delta(1600)^*$

- Treat $\Delta(1600)$ as radial excitation of the $\Delta(1236)$
- Calculate pion cloud contributions from the CBM using intermediate $N$, $N^*$, $\Delta$, and $\Delta^*$

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*Ramalho & Tsushima, PRD 82, 073007 (2010)*
Results: $\gamma^* + p \rightarrow S_{11}(1535)^*$

- Chose a radial wave function identical to the nucleon; only angular (P-wave) part is different. No free parameters
- Bare contribution is close to the bare contribution extracted from the EBAC analysis
- Meson cloud (pion+eta) is predicted to be large and of opposite sign

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*Ramalho & Pena, PRD 84, 033007 (2011)*
PART III:

Lessons from the study of DIS
The core wave functions can be determined from DIS (1)*

- DIS structure functions calculated from the handbag diagram

- Model nucleon wave functions depend on \( k = M \kappa \) only through the covariant variable

\[
\chi = \frac{2 P \cdot k}{M m_s} - 2 = 2 \sqrt{1 + \frac{\kappa^2}{r^2}} - 2 \quad r = \frac{m_s}{M}
\]

- For S-wave nucleons, the structure function is

\[
f_q(x) = \frac{M m_s \lambda^2}{16 \pi^2} \int_{\xi}^{\infty} d\chi \left[ \psi_q(\chi) \right]^2 \quad \text{with} \quad \xi = \frac{(r + x - 1)^2}{r(1 - x)}
\]

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*FG, Ramalho, Pena, PRC 77, 015202 (2008)

*FG, Ramalho, and Pena, PRD 85,093006 (2012)

FG, Ramalho, and Pena, PRD 85,093005 (2012)
The core wave functions can be determined from DIS (2)

- Our original choice (2008) using wave function fit to form factors

\[
\psi(\chi) = \frac{N}{(\beta_1 + \chi)(\beta_2 + \chi)}
\]
The core wave functions can be determined from DIS (2)

- Our original choice (2008) using wave function fit to form factors
  \[ \psi(\chi) = \frac{N}{(\beta_1 + \chi)(\beta_2 + \chi)} \]

- New choice (2012) using wave function fit directly to structure function
  \[ \psi(\chi) = N \frac{\beta \cos \theta + \chi \sin \theta}{\chi^\alpha (\beta + \chi)^{n-\alpha}} \]
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- Then add P and D-states
Solution 1 (18.5% P-state; 3.2% D-state): \( f(x) \) and \( g_1(x) \)
Prediction for $g_2(x)$

- Only Solution 2 (35% D state) gives a good result

![Graph showing predictions for $g_2(x)$ with two solutions: Solution 1 and Solution 2. Solution 2 is marked with a red dotted line, while Solution 1 is marked with a solid red line. The graph compares the predictions against experimental data.]
Discussion and Implications

- Existence and positions of “bare” N* poles are needed to construct the hadronic lagrangian -- the goal of this program (??)

- High $Q^2$ vertex functions determined from CST modeled N and N* wave functions unify our understanding of
  - experimental data at high $Q^2$
  - LQCD (at larger pion masses)
  - coupled channel calculations

- CST allows the nucleon wave function to be fixed directly from the experimental DIS data. The nucleon is expected to have a large D-state.

- To do:
  - Calculate pion cloud contributions within the CST framework
  - Fix the quark form factors by refitting the nucleon form factors using new DIS determined wave function and new (theoretical) pion cloud contributions
  - Develop a covariant, gauge invariant, coupled-channel scheme for fitting photo-production data that uses CST vertex functions, parameterized quark form factors, and meson interactions consistent with pion cloud calculations.
END
Theoretical issues to address when building a coupled channel SCHEME

★ What form of relativistic dynamics, and what degrees of freedom?
  • Hadronic d.o.f.: hamiltonian instant form (Sato and Lee)
  • Hadronic d.o.f.: manifestly covariant (Bethe-Salpeter or Covariant spectator theory)
  • Point-like (current) quark d.o.f.: front form (Brodsky)

★ Electromagnetic Gauge invariance (use Gross&Riska?)

★ Orthogonality of the $P_{11}$ states (including the nucleon)

★ Consistent treatment of
  • $s$ and $u$ channels,
  • two and three body final states
  • pion cloud physics

★ Beware! Interpretation of angular momentum depends on the form of dynamics and the degrees of freedom
Two differing pictures -- with equivalent physics

- Light front field theory (or quantum mechanics)
- Covariant Spectator Theory (CST)

How do they describe a “typical” QCD diagram? (For the $N \rightarrow \Delta$ transition, for example)
“Typical QCD diagram” : Light-Front interpretation
“Typical QCD diagram”: CST interpretation

Dressed quark form factor from vector dominance

3-quark wave function of the baryon
Angular momentum content depends on your point of view

Two ways to interpret this process

Light cone; point-like quarks

\[ \langle N | qqq \rangle \quad e_q \gamma^\mu \quad \langle qqq | N \rangle \]

CST; constituent quarks

\[ \langle N | qqq \rangle \quad e_q F(q) \gamma^\mu \quad \langle qqq | N \rangle \]
Spin-isospin structure (2): nucleon

- Spin-isospin structure of the NR nucleon wave function (cont’d)
  - Introduce a mathematically compact form; suppress name of scalar diquark
  \[
  |s_f\rangle = \frac{1}{\sqrt{2}} \left\{ \chi_s \chi_f - \frac{1}{3} \left[ \sigma \cdot \xi_m^* \chi_s \right] [\tau \cdot \xi_F^* \chi_f] \right\}
  \]
  (0,0) diquark
  scalar spin
  scalar flavor
  (1,1) diquark
  axial vector spin
  vector flavor

- Relativistic wave function, including spin-flavor structure
  \[
  \Psi_N(s_f) = \frac{1}{\sqrt{2}} \left\{ u(P,s) \chi_f^* + \frac{1}{3} \left( \gamma^s \xi_m^* \right) u(P,s) \left[ \xi_F^* \cdot \tau \chi_f^* \right] \right\} \phi(P,p_s)
  \]

- When \( P = 0 \), the lower component is 0 and this reduces exactly to the nonrelativistic form

This is a fixed-axis state
Spin-isospin structure (3): Delta

Spin-isospin structure of the Delta wave function is pure (1,1); diquark with axial-vector spin and vector flavor

\[ \Psi_\Delta (sf) = -\left[ \xi_F^* \cdot T \tilde{\chi}^f \right] \mathcal{E}_m^\mu \mathcal{w}_\mu (P, s) \phi (P, p_s) \]

- where \( \mathcal{E}_m^\mu \) is a fixed-axis axial-vector polarization
- \( \mathcal{w}_\mu (P, s) \) is a Rarita-Schwinger wave function satisfying
  \[ \gamma^\mu \mathcal{w}_\mu = 0; \quad P^\mu \mathcal{w}_\mu = 0 \]
- \( T^i \) is an isospin 3/2 -> 1/2 transition operator
- \( \tilde{\chi}^f \) is the isospin state of the \( \Delta \)

when \( P = 0 \), the lower component is zero and this reduces exactly to the nonrelativistic form
Relativistic impulse approximation for the form factors

In the spectator theory, the photon couples to the off-shell quark, and because of the symmetry, the coupling to all three quarks is 3 times the coupling to one

\[ J_\mu = \bar{u}(P_+, \lambda') \frac{3}{2} \int \frac{d^3 p_s}{(2\pi)^3 2E_s(p_s)} \phi(P_+, p_s) \phi(P_-, p_s) \left\{ j_\mu^I - \frac{1}{9} \gamma^\nu \gamma^5 \tau^j j_\mu^I \gamma^5 \gamma^\nu D_{\nu\nu'} \right\} u(P_-, \lambda) \]

integrate over the (on-shell) spectator three momentum

quark currents with form factors

sum over the vector diquark polarization

\[ D_{\nu\nu'} = \sum_\lambda \epsilon_\nu \epsilon_{\nu'}^* \]

approximate the two spectator quarks by a single diquark with a fixed mass

\[ P_\pm = P \pm \frac{1}{2} q \quad Q^2 = -q^2 \]

\[ \phi(P \cdot p) = \frac{N_0}{(\alpha_1 + \chi(P \cdot p))(\alpha_2 + \chi(P \cdot p))} \]
**Quark form factors (based on vector meson dominance)**

- **The quark currents are**
  \[ j_i^\mu = j_1 \left( \gamma^\mu - \frac{q^\mu}{q^2} \right) + j_2 \frac{i\sigma^{\mu\nu}q_\nu}{2M}, \]
  with 4 form factors:
  \[ f_i^\pm(0) = 1 \]
  \[ f_2^\pm(0) = \kappa_\pm \text{ quark anomalous moment} \]

- **The quark form factors come from vector dominance**

  \[ f(Q^2) = e + gB(Q^2)e + gB(Q^2)gB(Q^2)e + \cdots = e + \frac{gB(Q^2)e}{1 - gB(Q^2)} \]

  if \[ gB(Q^2) = \frac{\lambda^2}{\Lambda^2 + Q^2}, \text{ then } f(Q^2) = e + \frac{\lambda^2e}{\Lambda^2 - \lambda^2 + Q^2} \]

- **We use**

  \[ f_{1\pm} = \lambda + \frac{1 - \lambda}{1 + Q_0^2/m_v^2} + \frac{c_\pm Q_0^2/M_h^2}{(1 + Q_0^2/M_h^2)^2} \]
  \[ f_{2\pm} = \kappa_\pm \left( \frac{d_\pm}{1 + Q_0^2/m_v^2} + \frac{(1 - d_\pm)}{1 + Q_0^2/M_h^2} \right) \]

  3 parameters \( \lambda \) fixed by DIS
  4 parameters \( \kappa_\pm \) fixed by moments
  \( c_\pm \) fit
  \( d_\pm \) fit
The sum over the diquark polarization is tricky

★ First, must be in a **collinear frame** so that the fixed-axes (tied to the nucleon and Δ momentum) are identical

★ Then, the most general form is

\[
D^{\mu \nu} = \sum_{\lambda} \varepsilon_+^{\mu} \varepsilon_-^{\nu*} = \left( -g^{\mu \nu} + \frac{P_+^{\mu} P_+^{\nu}}{P_-^2} \right)
\]

\[
+ a_1 \left( P_- - \frac{b P_+}{M_+^2} \right)^{\mu} \left( P_+ - \frac{b P_-}{M_-^2} \right)^{\nu}
\]

**with** \( b = (P_+ \cdot P_-) \) **and** \( a = \frac{M_+ M_-}{b(M_+ M_- + b)} \)

★ For equal masses this becomes

\[
D^{\mu \nu} = -g^{\mu \nu} - \frac{P_+^{\mu} P_-^{\nu}}{M_-^2} + 2 \frac{P_+^{\mu} P_-^{\nu}}{P_-^2} \quad \text{where} \quad P = \frac{1}{2}(P_+ + P_-)
\]
Results: Nucleon form factors

★ Nucleon Form factors
★ Four models $\chi^2/N$

I (4 parameters)
$\alpha_1 \alpha_2$ quarks with isospin symmetry
$c_+ = c_-$ 9.26
$d_+ = d_-$

II (5 parameters)
$\alpha_1 \alpha_2$ break charge symmetry
$c_+ \neq c_-$ 1.36
$d_+ = d_-$ Best phenomenology!

III (6 parameters)
isospin symmetry with pion clouds
1.85

IV (9 parameters)
pion cloud and broken isospin symmetry
1.03
Quark distribution function from DIS

- Our model predicts the quark distribution amplitude measured in DIS
- Our normalization gives
  \[ 1 = \int_{-\infty}^{1} dx f(x) \text{ instead of } 1 = \int_{0}^{1} dx f(x) \]
- Choose quark charge at \( Q^2 = \infty \) to be \( \lambda > 1 \), where
  \[ \int_{0}^{1} dx f(x) = \frac{1}{\lambda^2} < 1 \]
- Choose diquark mass to give experimental momentum fraction
  \[ \frac{\langle x f \rangle}{\langle f \rangle} = \frac{\int_{0}^{1} dx \, x f(x)}{\int_{0}^{1} dx f(x)} = 0.171 \]
The core wave functions can be determined from DIS*

- DIS structure functions calculated from the handbag diagram
- Model nucleon wave functions depend on $k = M \kappa$ only through the covariant variable
  \[
  \chi = \frac{2P \cdot k}{Mm_s} - 2 = 2\sqrt{1 + \frac{\kappa^2}{r^2} - 2} \quad r = \frac{m_s}{M}
  \]
- In DIS limit, $x$ dependence emerges naturally. We chose $m_s = M$ to map $k = 0$ to $x = 0$
  \[
  \kappa \geq |\kappa_{\text{min}}| \quad \kappa_{\text{min}} \equiv \frac{r^2 - (1 - x)^2}{2(1 - x)}
  \]
- Choose functional form of wave function and adjust parameters to fit $xf(x)$
- DIS choice is
  \[
  \psi(\chi) = N \frac{\beta \cos \theta + \chi \sin \theta}{\chi^\alpha (\beta + \chi)^{n-\alpha}}
  \]

*FG, Ramalho, and Pena, PRD 85,093006 (2012)
FG, Ramalho, and Pena, PRD 85,093005 (2012)