Into. To Lattice QCD

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Outline

- Introduction to QCD and path integral
- Lattice, scalar field
- Gauge field
- Lattice fermion
- Quenched approximation
- Some results

QCD

- Quantum field theory of strong interaction
- Color field group SU(3)
- Different from QED because of gluon-gluon interactions
- At high energies:
  - small coupling constant
  - perturbation theory can apply
- At low energies:
  - large coupling constant
  - perturbation theory does not apply

QCD Lagrangian

$$L_{QCD} = -\frac{1}{4} G_a^{\mu\nu} G_{\mu\nu}^a + \sum_f \bar{q}_f [i\gamma^\mu D_\mu - m_f] q_f$$

with the gluon field strength tensor

$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g f_a^{bc} A_b^\mu A_c^\nu$$

and the gauge covariant derivative

$$D^\mu = \partial^\mu - i\frac{g}{2} A_a^\mu \lambda^a$$

where A_a^μ is the gluon field, g is the strong coupling constant and f denotes the quark flavor. Looks very similar to QED, except for the last term in the second equation.

Path Integral

- Expectation value of an observable

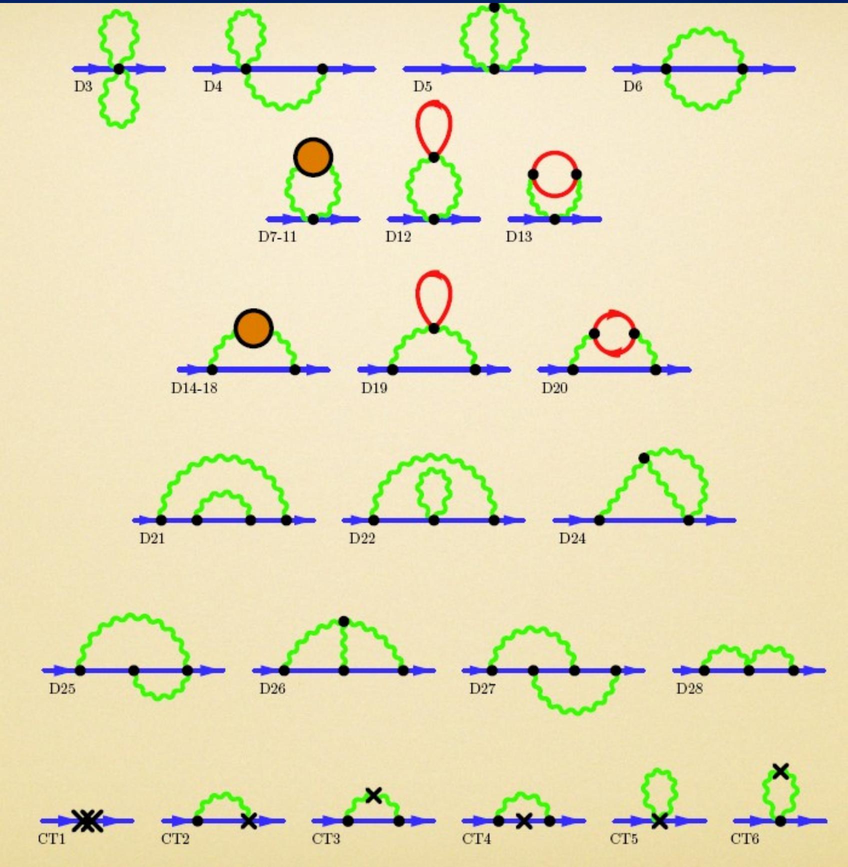
$$\langle O \rangle = \frac{1}{Z} \int d\psi \, d\overline{\psi} \, dA \, O e^{iS}$$

$$Z = \int d\psi \, d\overline{\psi} \, dA \, e^{iS}$$

$$S = \int dx \, L$$

Perturbation Theory

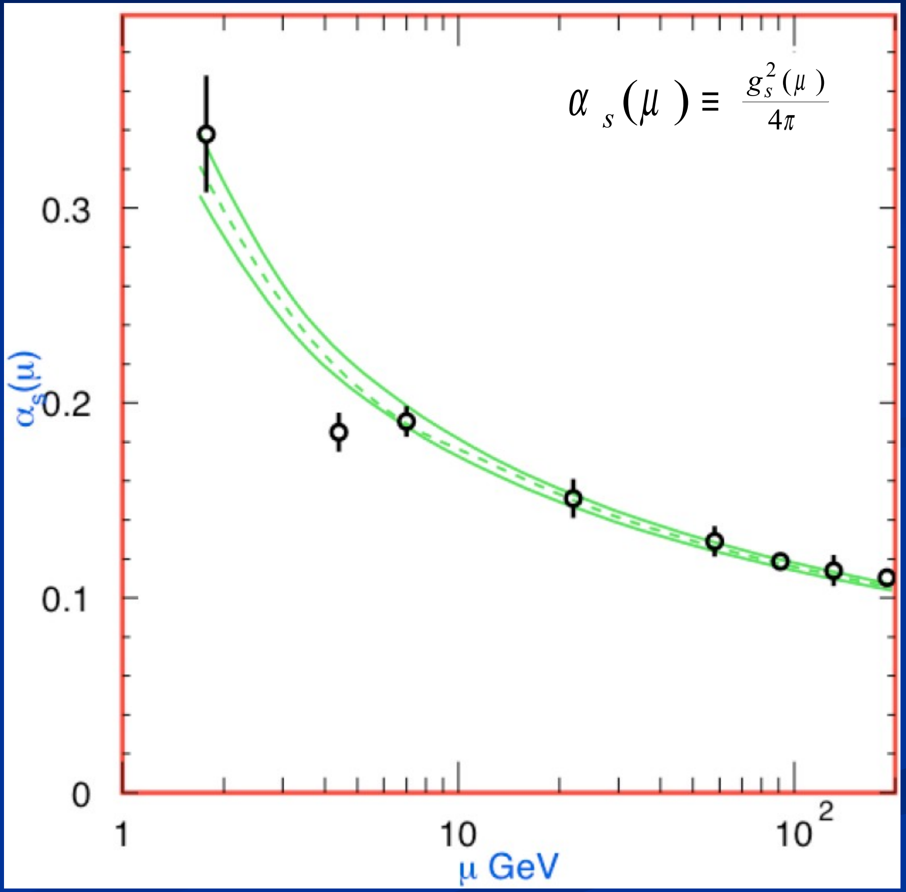
- Calculate Feynman diagrams.
- Stop at certain order.
- Order corresponds to number of vertices.
- Proportional to coupling constant, only applicable for small coupling constant.



I. Allison, “Matching the Bare and *MS* Charm Quark Mass using Weak Coupling Simulations”, presentation at *Lattice 2008*

Intrinsic QCD Scale

- Running coupling constant.
- Intrinsic QCD scale Λ_{QCD} in the order of 1 GeV.
- Scale below which the coupling constant becomes so large that standard perturbation theory no longer applies.
- Many unresolved question about low-energy QCD.
- This is where Lattice QCD comes in!



R. Timmermans, D. Bettoni and K. Peters, "Strong interaction studies with antiprotons"

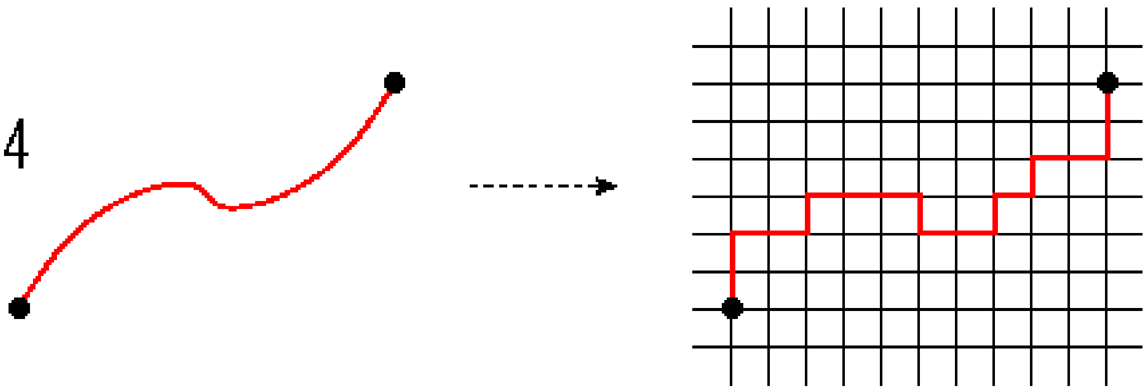
Lattice QCD

- Proposed by Wilson, 1974.
- Nonperturbative low-energy solution of QCD.
- E.O.M. discretized on 4d Euclidean space-time lattice.
- Quarks and gluons can only exist on lattice points and travel over connection lines.
- Solved by large scale numerical simulations on supercomputers.

Lattice

$$x_0 = -ix_4$$

$$S_E = iS_M$$

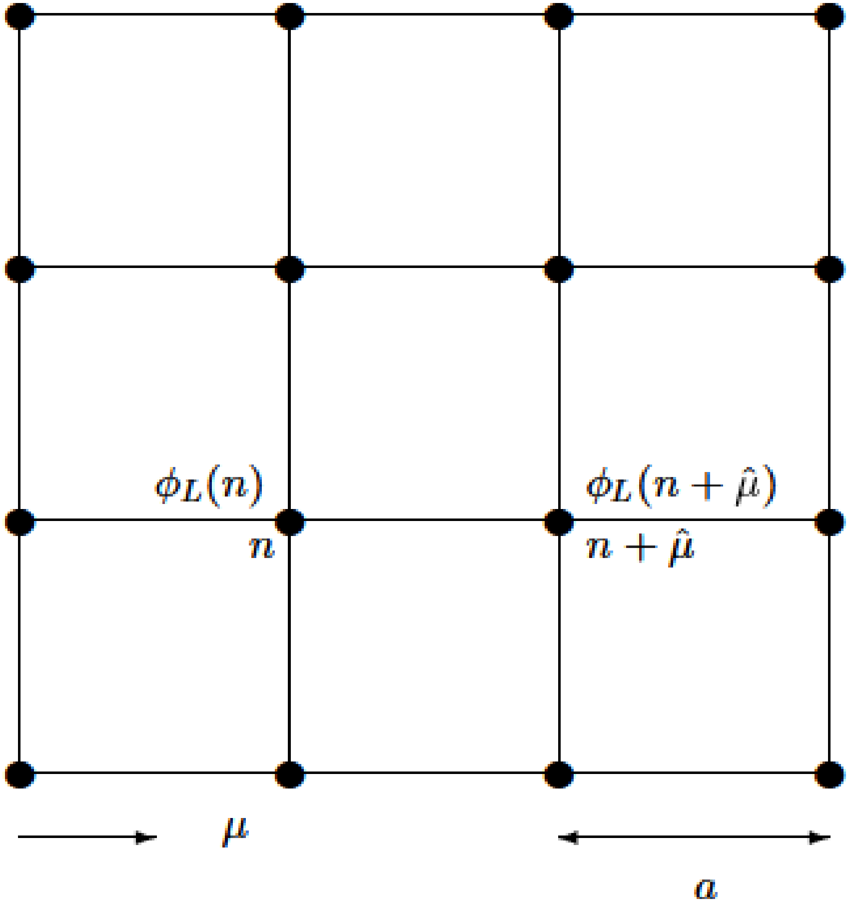


- From continuum to discretized lattice:

$$\int d^4x \to a^4 \sum_n$$

- n four-vector that labels the lattice site, a lattice constant
- Check, take an appropriate continuum limit ($a \to 0$) to get back the continuum theory.

Scalar Fields on a lattice



Scalar field lives on a lattice site

$$\phi(x) \rightarrow \phi(na)$$

$$n = (n_1, n_2, \cdots, n_D)$$

Derivative

$$\partial_\mu \phi(x) \rightarrow$$

$$\frac{\phi((n+\hat{\mu})a) - \phi((n-\hat{\mu})a)}{2a}$$

Dimensionless field

$$\phi_L(n) = a\phi(na)$$

Scalar field action

- Scalar field $\Phi(x)$, action of continuum field theory in Euclidean space:
  $$S(\Phi) = \int d^4x[\tfrac{1}{2}(\partial_\mu \Phi)^2 + \tfrac{1}{2}m^2\Phi^2 + \tfrac{\lambda}{4}\Phi^4]$$
- Discretize to a lattice:
  $$\int d^4x \to a^4\sum_n$$
  $$\Phi(x) \to \Phi_n$$
  $$\partial_\mu \Phi(x) \to \tfrac{1}{a}(\Phi_{n+\hat{\mu}} - \Phi_n)$$
- Result:
  $$S(\Phi) = \sum_n \left\{\tfrac{a^2}{2}\sum_{\mu=1}^{4}(\Phi_{n+\hat{\mu}} - \Phi_n)^2 + a^4(\tfrac{m^2}{2}\Phi_n^2 + \tfrac{\lambda}{4}\Phi_n^4)\right\}$$

Expectation value calculation

- Feynman path integral formalism
- Expectation value of an operator

$$< 0 | O(\Phi_{n_1}, \Phi_{n_2}, \ldots, \Phi_{n_l}) | 0 > = \frac{1}{Z} \int \prod_n [d\Phi_n] O(\Phi_{n_1}, \Phi_{n_2}, \ldots, \Phi_{n_l}) e^{-S(\Phi)}$$

where

$$Z = \int \prod_n [d\Phi_n] e^{-S(\Phi)}$$

- Rescale fields: $\Phi'_n = \sqrt{\lambda}\, \Phi_n$
- Lattice action becomes: $S(\Phi) = \frac{1}{\lambda} S'(\Phi')$

Statistical Mechanics

- Rescaled expectation value

$$< 0 | O(\Phi'_{n_1} \Phi'_{n_2} \ldots \Phi'_{n_l}) | 0 > = \tfrac{1}{Z'} \int \prod_n [d\Phi'_n] O(\Phi'_{n_1} \Phi'_{n_2} \ldots \Phi'_{n_l}) \exp\{- \tfrac{1}{\lambda} S'(\Phi')\}$$

$$Z' = \int \prod_n [d\Phi'_n] \exp\{- \tfrac{1}{\lambda} S'(\Phi')\}$$

- Recognizable?
- Canonical ensemble

$$\frac{1}{\lambda} \to \beta \equiv \frac{1}{kT}$$

- Similar for fermion fields

Statistical Mechanics

The equivalences between a Euclidean field theory and Classical Statistical Mechanics.

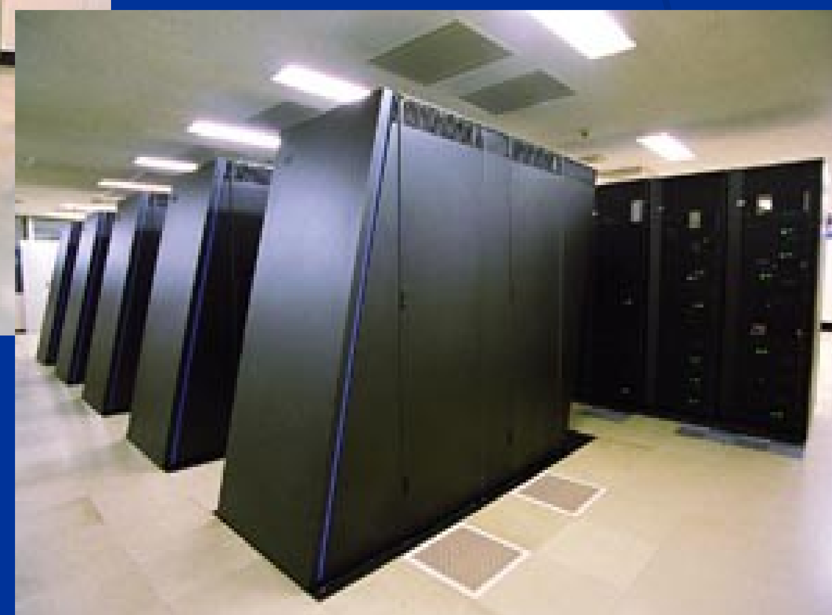
Euclidean Field Theory	Classical Statistical Mechanics
Action	Hamiltonian
unit of action h	units of energy $\beta = 1/kT$
Feynman weight for amplitudes $e^{-\mathcal{S}/h} = e^{-\int \mathcal{L} dt/h}$	Boltzmann factor $e^{-\beta H}$
Vacuum to vacuum amplitude $\int \mathcal{D}\phi e^{-\mathcal{S}/h}$	Partition function $\sum_{conf.} e^{-\beta H}$
Vacuum energy	Free Energy
Vacuum expectation value $\langle 0 \vert \mathcal{O} \vert 0 \rangle$	Canonical ensemble average $\langle \mathcal{O} \rangle$
Time ordered products	Ordinary products
Green's functions $\langle 0 \vert T[\mathcal{O}_1 \ldots \mathcal{O}_n] \vert 0 \rangle$	Correlation functions $\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle$
Mass M	correlation length $\xi = 1/M$
Mass-gap	exponential decrease of correlation functions
Mass-less excitations	spin waves
Regularization: cutoff Λ	lattice spacing a
Renormalization: $\Lambda \to \infty$	continuum limit $a \to 0$
Changes in the vacuum	phase transitions

R. Gupta, "Introduction to Lattice QCD", arXiv:hep-lat/9807028

Monte Carlo Method

- Method from statistical mechanics to calculate expectation value numerically.
- Generate random distribution.
- Calculate expectation value for this distribution.
- Repeat this process very many times.
- Average over results.
- Results have statistical errors.
- A lot of computational power needed!

Supercomputers



http://www.kek.jp/intra-e/press/2006/supercomputer_e.html

Gauge Fields on a lattice

Scalar ⇨ site n ⟺ Vector ⇨ link (n, μ)

(continuum) Gauge Fields $A_\mu(x) = A_\mu^a(x) T^a \in \mathcal{SU}(N)$

$$\mathrm{tr}\, T^a = 0, \ (T^a)^\dagger = T^a \Rightarrow \quad T^a : \ \text{Traceless, Hermite}$$

$$\mathrm{tr}\, T^a T^b = \frac{1}{2}\delta^{ab} \quad \text{orthogonality, normalization}$$

$$[T^a, T^b] = i f^{abc} T^c, \quad f^{abc} \text{ structure constant of group SU(N)}$$

Problem: A_μ is not gauge covariant !

$\Omega(x) \in SU(N)$: gauge tr.

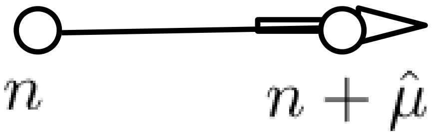
$$A_\mu(x) \to \frac{\Omega(x)\partial_\mu \Omega^\dagger(x)}{ig} + \Omega(x) A_\mu(x) \Omega^\dagger(x)$$

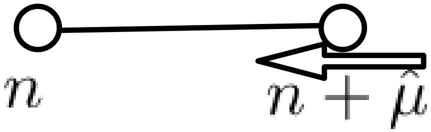
Gauge Fields on a lattice II

g : coupling constant

Link variables

$$U_{n,\mu} = \exp[igaA_\mu(n)] \in \ \mathrm{SU(N)}$$





$$U_{n+\hat{\mu},-\mu} \equiv U^\dagger_{n,\mu}$$

Gauge transformation

$$g_n \in \ \mathrm{SU(N)}$$

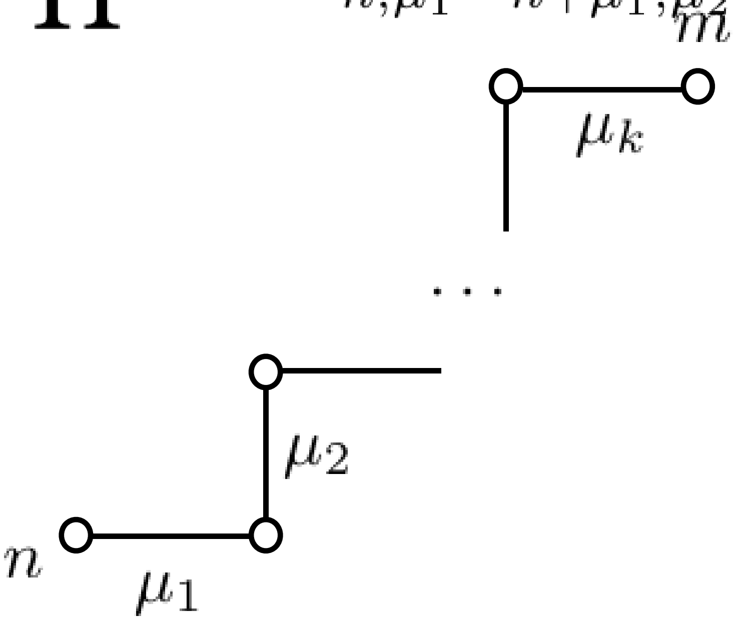
$$U_{n,\mu} \rightarrow U^g_{n,\mu} = g_n U_{n,\mu} g^\dagger_{n+\hat{\mu}}$$

covariant

Gauge Invariance III

Product of link variables

$$\prod U \equiv U_{n,\mu_1} U_{n+\hat{\mu}_1,\mu_2} \cdots U_{m-\hat{\mu}_k,\mu_k} \longrightarrow g_n \prod U g_m^\dagger$$



Closed loop C at n

$$\prod_C U \to g_n \left\{ \prod_C U \right\} g_n^\dagger$$

$$\Rightarrow \quad \mathrm{tr} \left\{ \prod_C U \right\} \text{ is gauge invariant}$$

Wilson Loops

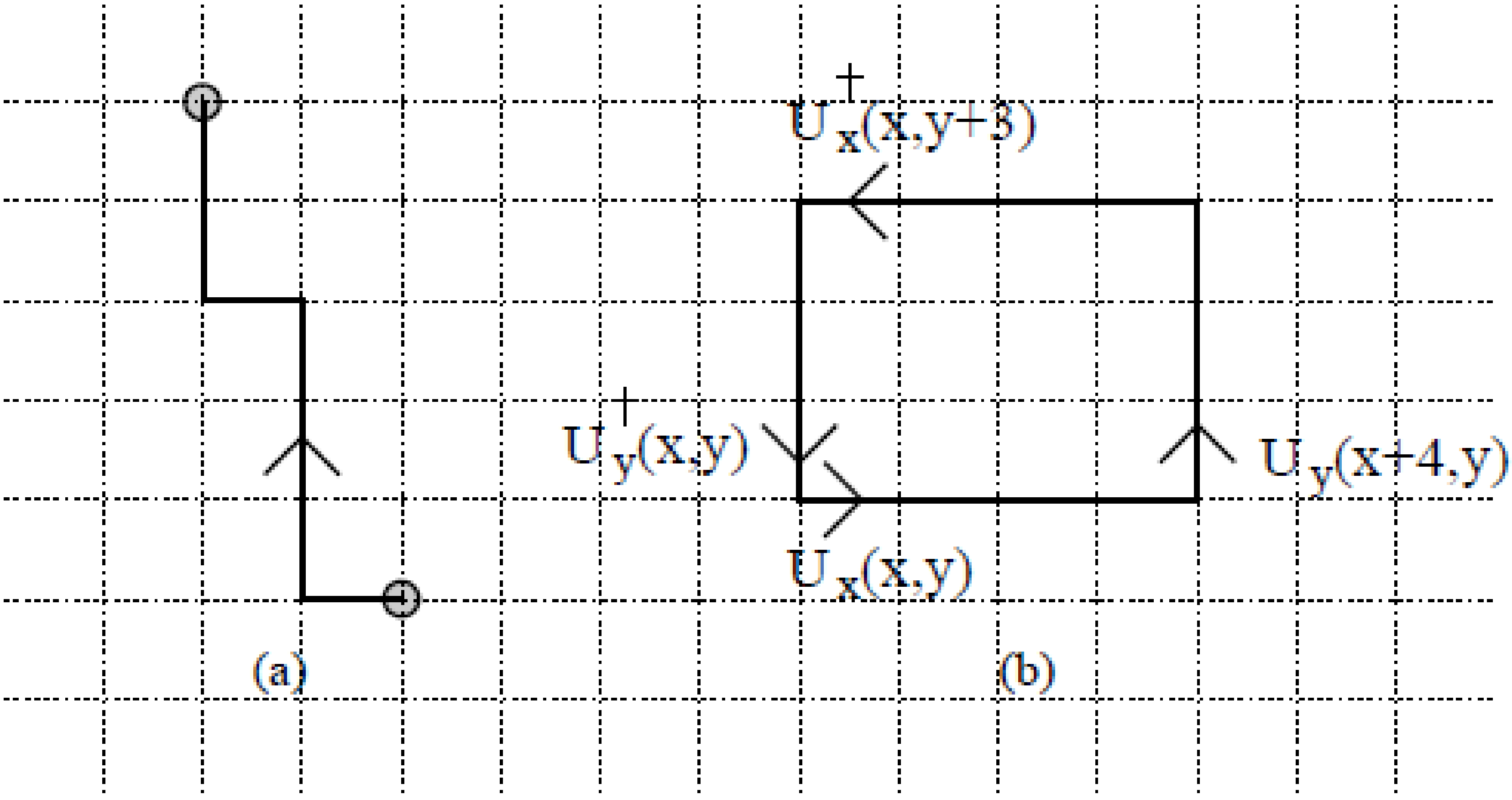


Fig. 5. The two gauge invariant quantities. a) An ordered string of $U's$ capped by a fermion and an anti-fermion and b) closed Wilson loops.

R. Gupta, "Introduction to Lattice QCD", arXiv:hep-lat/9807028

Wilson action

$$S_W = \frac{1}{g^2} \mathrm{Re} \sum_{x,\mu>\nu} \mathrm{Tr}\, \tfrac{1}{2}\left(1 - U_{x,\mu} U_{x+a\hat{\mu},\nu} U^{\dagger}_{x+a\hat{\nu},\mu} U^{\dagger}_{x,\nu}\right)$$

- Simplest discretized action of the Yang-Mills part of the QCD action
- Agrees with the QCD action to order O(a_2).
- Proportional to the gauge-invariant trace of the sum over all plaquettes.

From Wilson to Yang Mills

- Matrices U given by $U_\mu(x) = \exp(iagA_\mu(x+\frac{\hat{\mu}}{2}))$
- The simplest Wilson loop, the 1x1 plaquette given by

$$W_{\mu\nu}^{1x1} = U_\mu(x)U_\nu(x+\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(x)$$

$$= \exp(iag[A_\mu(x+\tfrac{\hat{\mu}}{2}) + A_\nu(x+\hat{\mu}+\tfrac{\hat{\nu}}{2}) - A_\mu(x+\hat{\nu}+\tfrac{\hat{\mu}}{2}) - A_\nu(x+\tfrac{\hat{\nu}}{2})])$$

- Expanding about $x+\frac{\hat{\mu}+\hat{\nu}}{2}$ gives

$$= \exp[ia^2g(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{ia^4g}{12}(\partial_\mu^3 A_\nu - \partial_\nu^3 A_\mu) + \ldots]$$

- The Taylor series of the exponent gives

$$= 1 + ia^2gF_{\mu\nu} - \frac{a^4g^2}{2}F_{\mu\nu}F^{\mu\nu} + O(a^6) + \ldots$$

- From this we derive

$$\mathrm{Re}\,\mathrm{Tr}(1 - W_{\mu\nu}^{1x1}) = \frac{a^4g^2}{2}F_{\mu\nu}F^{\mu\nu} + \ldots$$

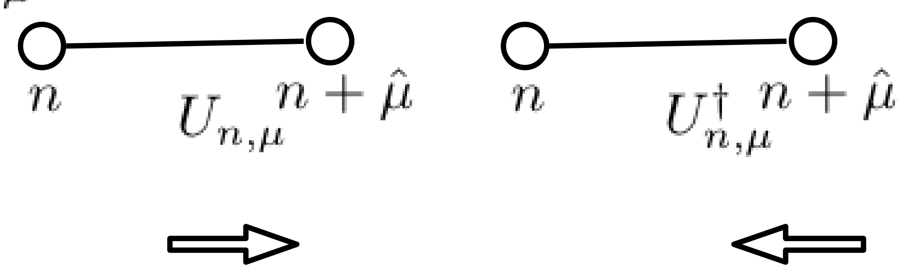
Lattice Fermions: Naive Fermion

$$S_F = \int d^4x\, \bar{\psi}(\gamma_\mu D_\mu + m)\psi$$

$$S_F = a^4 \sum_n \bar{\psi}_n \left[\sum_\mu \gamma_\mu \frac{U_{n,\mu}\psi_{n+\hat{\mu}} - U^\dagger_{n-\hat{\mu},\mu}\psi_{n-\hat{\mu}}}{2a} + m\psi_n \right]$$

$$a^{3/2}\psi \to \psi,\ ma = M\ (\text{ dimensionless})$$

$$S_F = \frac{1}{2}\sum_{n,\mu}\left[\bar{\psi}_n\gamma_\mu U_{n,\mu}\psi_{n+\hat{\mu}} - \bar{\psi}_{n+\hat{\mu}}U^\dagger_{n,\mu}\psi_n\right] + M\bar{\psi}_n\psi_n$$



Fermion doubling

$$g^2 \to 0 \ ({}^{\forall}U_{n,\mu} = 1) \qquad S_F = \int \frac{d^4p}{(2\pi)^4} \bar{\psi}(-p) \left[i\gamma_\mu \sin(p_\mu a) + M \right] \psi(p)$$

propagator

$$G_F(p) = \frac{1}{i\gamma \cdot s + M} = \frac{-i\gamma \cdot s + M}{s^2 + M^2}$$

$$i\gamma \cdot s = \gamma_\mu s_\mu = \gamma_\mu \sin(p_\mu a)$$

pole of $G_F(p)$ in $a \to 0$ ($\hat{p}_\mu \ll 1/a$)

$$\sin(p_\mu a) = \begin{cases} \hat{p}_\mu a & p_\mu = \hat{p}_\mu \\ -\hat{p}_\mu a & p_\mu = \pi/a + \hat{p}_\mu \end{cases}$$

$$\Rightarrow \lim_{a \to 0} G_F(p) = \frac{1}{a} \sum_{p_\mu = 0, \pi/a} \frac{-i(-1)^\delta \gamma \cdot \hat{p} + m}{\hat{p}^2 + m^2}$$

$\delta_\mu = 0$ for $p_\mu = 0$

$\delta_\mu = 1$ for $p_\mu = (\pi/a)$

1 lattice fermion field $\Rightarrow 2^d = 16$ particles ("doubling problem")

$$2^d \to \begin{cases} 2^{d-1} & \text{chirality } + & (|\delta| = \text{ even}) \\ 2^{d-1} & \text{chirality } - & (|\delta| = \text{ odd}) \end{cases}$$

Solution: Wilson fermions

Add $O(a)$ term $\Rightarrow$ mass to doublers $({}^{\exists}p_\mu = \pi/a)$ "Wilson term"

$$S_W = -ar\int d^4x\,\bar\psi D^2\psi \;\rightarrow -\frac{r}{2}\sum_{n,\mu}\left[\bar\psi_n U_{n,\mu}\psi_{n+\hat\mu} + \bar\psi_{n+\hat\mu}U^\dagger_{n,\mu}\psi_n - 2\bar\psi_n\psi_n\right]$$

$$S_F = S_F^0 + S_W \longrightarrow \bar\psi(\gamma\cdot D + m)\psi \quad (a\rightarrow 0)$$

$$g^2 = 0 (U_{n,\mu} = 1)$$

$$S_F = \bar\psi(-p)[i\gamma\cdot s + \overbrace{M + r\sum_\mu(1-\cos(p_\mu a)}^{M(p)})]\psi(p) \;\Rightarrow G_F(p) = \frac{-i\gamma\cdot s + M(p)}{s^2 + M(p)^2}$$

$$a \longrightarrow 0$$

$$M(p) = \begin{cases} ma & \text{for physical pole} \\ ma + 2r|\delta| & \text{for doublers}\end{cases} \Rightarrow \begin{cases} m_{\rm phys} & = & m \\ m_{\rm doubler} & = & m + \frac{2r}{a}|\delta| \rightarrow \infty\end{cases}$$

"decoupling of doublers at low energy"

Caution: Wilson term violates chiral symmetry

Method of operation

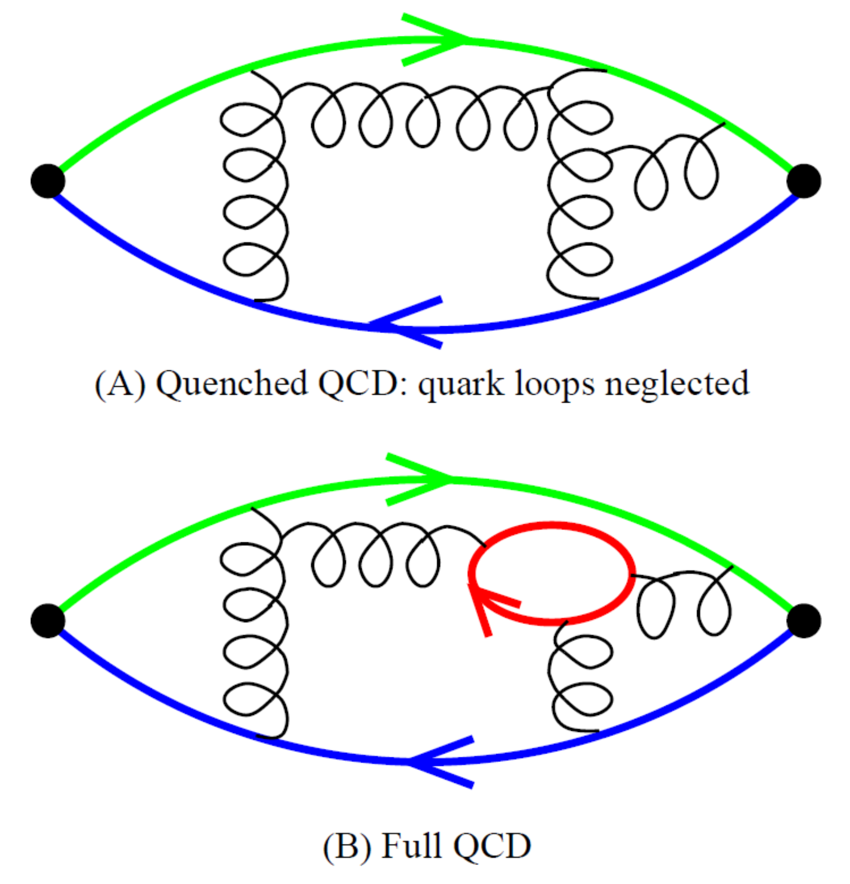
- Six unknown input parameters, coupling constant and the masses of the up, down, strange, charm and bottom quark.
- Top quark too short lived to form bound states at the energies we are looking at.
- Fix in terms of six precisely measured masses of hadrons.
- Masses and properties of all the other hadrons can be obtained this way.
- They should agree with experiment.

Lattice constant

- Lattice constant *a* should be small to approach continuum limit, but not too small or the computation time becomes too long.
- Size nucleon in the order of 1 Fermi (1 Fermi = 1.0×10^{-15} m).
- *a* between 0.05 and 0.2 Fermi
- Results also have systematic errors due to this lattice discretization.

Quenched Approximation

- Quarks fully dynamical degrees of freedom that can be produced and annihilated.
- In the quenched approximation vacuum polarization effects of quark loops are turned off.
- Very popular approximation, reduces computation time by a factor of about 10^3-10^5.



R. Gupta, "Introduction to Lattice QCD", arXiv:hep-lat/9807028

An Example: The Pion

- Calculate the Correlation Function

$$O = \left(\overline{\psi}(x)\gamma_5\psi(x)\right)^{+}\left(\overline{\psi}(0)\gamma_5\psi(0)\right)$$

- This should behave like

$$C(t) = \sum_n A_n e^{-m_n t}$$

- We want to find the ground state mass

$$C(t) \rightarrow \mathrm{A}_0 e^{-m_0 t}$$

As t becomes large

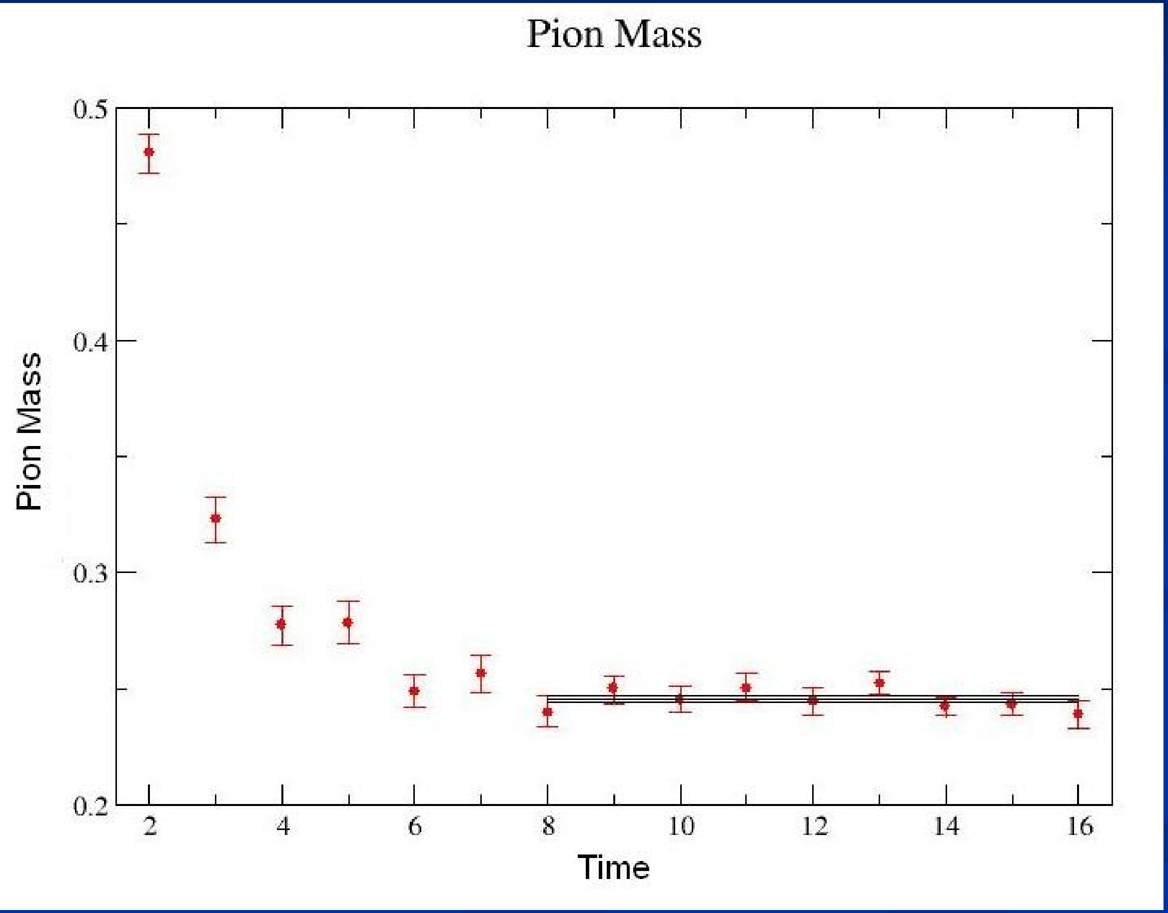
Mass of the ground state

- Plot $\ln\left(\frac{C(t)}{C(t+1)}\right)$ against t as

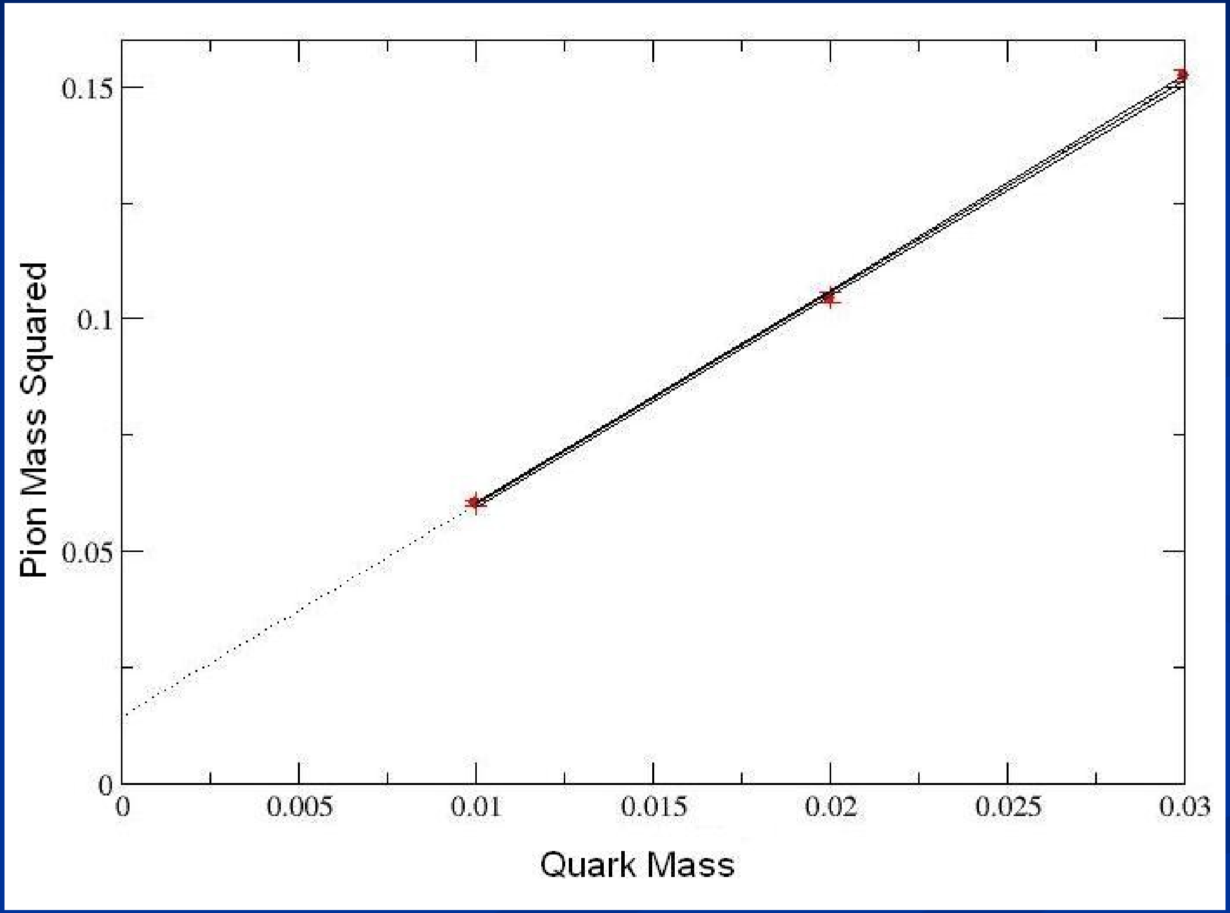
$$\ln\left(\frac{C(t)}{C(t+1)}\right) = \ln\left(\frac{A_0\, e^{-m_0 t}}{A_0\, e^{-m_0 (t+1)}}\right) = \ln\left(e^{m_0}\right) = m_0$$

- Look for a plateau

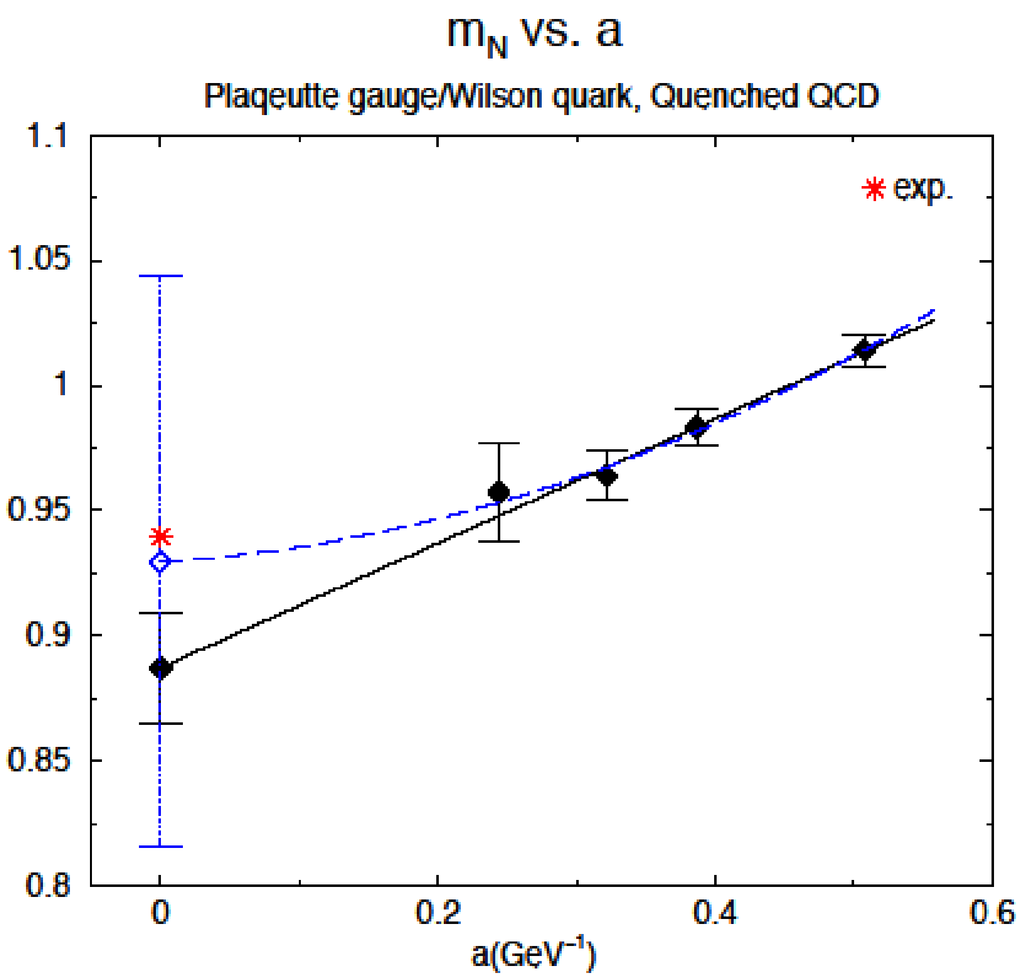
Mass Plot



Chiral Extrapolation

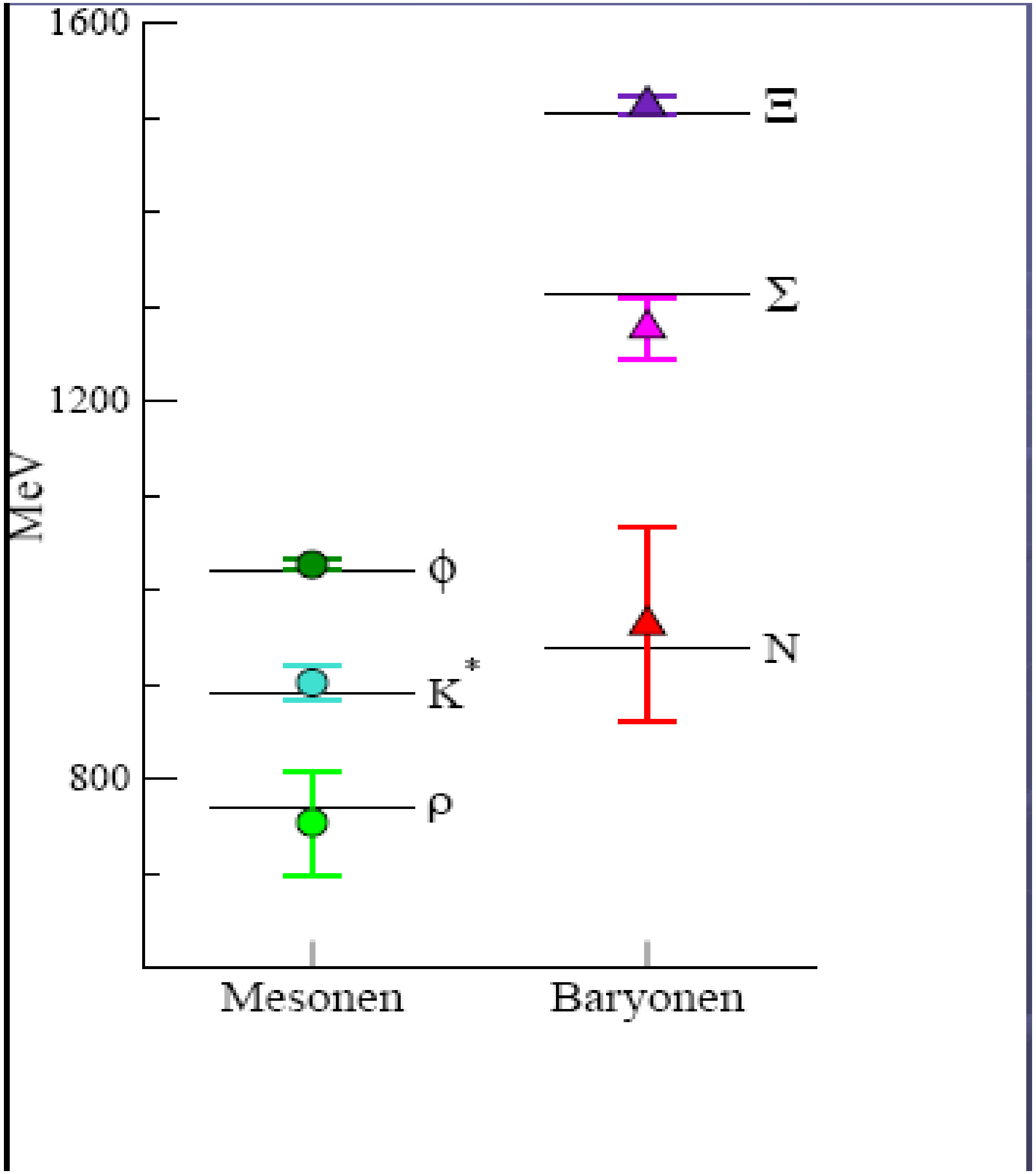


Nucleon Mass



Lattice spacing

$$m_N(a) = m_N(0) + C_1 a$$
$$m_N(a) = m_N + C_1 a^2 + C_2 a^2$$



1600
1200
800
MeV
Ξ
Σ
N
φ
K*
ρ
Mesonen
Baryonen

References

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Lattice QCD (introduction) by Polikarkov DUBNA WINTER SCHOOL 1 2 FEBRUARY 2005