Ostrogradsij's theorem

Implications on modified gravity theories

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Mikhail Ostrogradskij

- September 24, 1801 - January 1, 1862
- Mathematician, mechanician and physicist
- Did not realize the importance of his work

Overview

1. Why modified theories of gravity?
2. Ostrogradskij's theorem
3. Horndeski Theories
4. Beyond Horndeski

What is Einstein's theory of gravity?

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- **Unchanged** since its formulation
- Very well **tested** in numerous experiments
- It is the gravity theory used for ΛCDM **Cosmology**
- Field Theory (admits **Lagrangian Formulation**)

Why search for alternative theories?

ΛCDM paradigm assumes **infation**:

- Dark Energy ($\sim$ 96% of energy has so far only been detected gravitationally!)
- Fine tuning

Easy solution: a field theory that doesn't require Dark Energy!
Actually, it's not so easy.

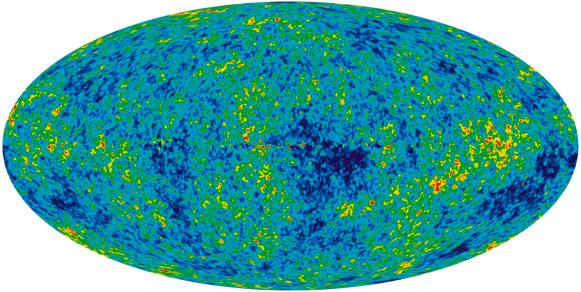

Figure: CMB temperature fluctuations acquired with WMAP

Classical case

Classically we deal with $L = L(q, \dot{q})$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

If the Lagrangian is nondegenerate and we define $Q := q$ and $P := \frac{\partial L}{\partial \dot{q}}$

$$H(Q, P) = Pv(Q, P) - L(Q, v(Q, P))$$

When the Lagrangian has no explicit time dependence, H is also the associated conserved quantity (i.e. **energy**)

Higher Derivatives

Higher derivative dependence: $L = L(q, \dot{q}, \ddot{q})$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0$$

$$q(t) = \mathcal{Q}(t, q_0, \dot{q}_0, \ddot{q}_0, \dddot{q}_0)$$

Four pieces of *initial value data* means four *canonical coordinates*:

$$Q_1 := q \qquad P_1 := \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}$$

$$Q_2 := \dot{q} \qquad P_2 := \frac{\partial L}{\partial \ddot{q}}$$

Higher Derivatives

Nondegeneracy implies that we can invert phase space transformation to solve for $\ddot{q}$

$$\exists \quad a(Q_1, Q_2, P_1) : \qquad \frac{\partial L}{\partial \ddot{q}}\Big|_{\ddot{q}=a} = P_2$$

$$H(Q_1, Q_2, P_1, P_2) = P_1 Q_2 + P_2 a(Q_1, Q_2, P_2) - L(Q_1, Q_2, a(Q_1, Q_2, P_2))$$

Again, when the Lagrangian has no explicit dependence on time, this is the energy of the system, but this time it is **linear** in P_1!

Example: the free particle

$$L = \frac{\dot{q}^2}{2} + \frac{\ddot{q}^2}{2}$$

$$\begin{aligned} Q_1 &= q & P_1 &= \dot{q} - \dddot{q} \\ Q_2 &= \dot{q} & P_2 &\ddot{q} \end{aligned}$$

$$H(Q_1, Q_2, P_1, P_2) = \frac{P_2^2}{2} + P_1 Q_2 - \frac{Q_2^2}{2}$$

$\Rightarrow$ "energy" depends linearly on P_1

It is straightforward to prove that adding higher order terms in the Lagrangian makes things worse!

General Case

Theorem (Ostrogradskij)

Every system described by a nondegenerate Lagrangian which depends upon more than one time derivative (in such a way that the dependence cannot be eliminated by partial integration) is instable.

Only way out (so far): **violating the nondegeneracy assumption**

Conventions

In the following, I will use the standard convention:

$$g_{ij,k} := \frac{\partial g_{ij}}{\partial x^k}$$

$$g_{ij,kh} := \frac{\partial g_{ij}}{\partial x^k \partial x^h}$$

Ricci tensor $R_{ij} = R^h_{ihj}$

Ricci scalar $R = R_{ij} g^{ij}$

Horndeski's theory ('72)

$$\mathcal{L} = \mathcal{L}(g_{ij}; g_{ij,k}; g_{ij,kh}; \phi; \phi_{,i})$$

Generate the following Euler-Lagrange equations

$$E^{ij}(\mathcal{L}) = \frac{\partial}{\partial x^k}\left[\frac{\partial \mathcal{L}}{\partial g_{ij,k}} - \frac{\partial}{\partial x^h}\left(\frac{\partial \mathcal{L}}{\partial g_{ij,hk}}\right)\right] - \frac{\partial \mathcal{L}}{\partial g_{ij}}$$

$$E(\mathcal{L}) = \frac{\partial}{\partial x^k}\left[\frac{\partial \mathcal{L}}{\partial \phi_{,k}}\right] - \frac{\partial \mathcal{L}}{\partial \phi}$$

- Can suffer Ostrogradskij instability
- Find constraints such that field equations are second order

Horndeski's theory ('72)

$$\mathcal{L} = \beta_1 \mathcal{L}_1 + \beta_2 \mathcal{L}_2 + \beta_3 \mathcal{L}_3 + \eta \mathcal{L}_4 + c \mathcal{L}_5$$

$$\mathcal{L}_1 = \sqrt{g}(R^2 - 4R_{ij}R^{ij} + R_{ijkl}R^{ijkl}), \qquad \mathcal{L}_2 = \sqrt{g}G^{ij}\phi_{,i}\phi_{,j}$$

$$\mathcal{L}_3 = \sqrt{g}R, \qquad \mathcal{L}_4 = \sqrt{g}, \qquad \mathcal{L}_5 = \sqrt{g} * R^{ij}_{kh}R^{kh}_{ij}$$

- E^{ij} and E are at most second order in g_{ij} and ϕ
- **Does not suffer from instability**

Horndeski's theory ('74)

- Previous treatment covered only a **restricted number of cases**
- What happens for a general Lagrangian density?

$$\mathcal{L} = \mathcal{L}(g_{ij}; g_{ij,i_1}; \ldots; g_{ij,i_1\ldots i_p}; \phi; \phi_{,i_1}; \ldots; \phi_{,i_1\ldots i_q})$$

- Keep the requirement for the **E-L equations to be at most of second order**

$$\Rightarrow \mathcal{L} = \mathcal{L}(g_{ij}; g_{ij,i_1}; g_{ij,i_1 i_2}; \phi; \phi_{,i_1}; \phi_{,i_1 i_2})$$

Horndeski theories

GENERALIZED GALILEONS:

$$\mathcal{L} = G_2(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi, X)R + G_{4X}[(\Box\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu}] +$$
$$G_5(\phi, X)G^{\mu\nu}\phi_{\mu\nu} - \frac{G_{5X}}{6}[(\Box\phi)^3 - 3\Box\phi\phi^{\mu\nu}\phi_{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\lambda}\phi^{\mu}_{\lambda}]$$

Where

$$X := \nabla_\mu\phi\nabla^\mu\phi \qquad \phi_{\mu\nu} := \nabla_\mu\nabla_\nu\phi$$

BUT, there is more...

Beyond Horndeski

Main problem is Ostrogradskij, NOT the order of E-L equations

$$L = \frac{a}{2}\ddot{\phi}^2 + b\ddot{\phi}\dot{q} + \frac{c}{2}\dot{q}^2 + \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi^2 - \frac{1}{2}\phi^2$$

- Higher order E-L equations
- Ostrogradskij ghosts

However, if kinetic matrix is degenerate, the system is **free from Ostrogradskij ghosts!**
$\Rightarrow$ **DHOST Theories**

References

R.P. Woodard
Avoiding Dark Energy with 1/R Modifications of Gravity

G. W. Horndeski, D. Lovelock
Scalar Tensor Field Theories (1972) Tensor, 24, 79

G.W. Horndeski
Second Order Scalar Tensor Field Equations in a Four Dimensional Space (1974) Int Journal Of Theoretical Physics, 10, 6

T. Clifton, P.G. Ferreira, A. Padilla, C. Skordis
Modified Gravity and Cosmology

T. Kobayashi
Horndeski theory and beyond: a review arXiv:1901.07183