

# Form Factors and Charge Density Distributions

- $F(\vec{k}) = \int d^3r \rho(\vec{r}) e^{i\vec{k}\vec{r}}$
- $\rho(\vec{r}) = |\Psi(\vec{r})|^2 = \Psi^*(\vec{r})\Psi(\vec{r})$
- $\Psi(\vec{r}) = \frac{1}{\sqrt{2\pi^3}} \int d^3\tilde{p} e^{-i\tilde{p}\vec{r}} \tilde{\Psi}(\tilde{p}) = \mathcal{F}\tilde{\Psi}$
- $\Psi(\vec{p}) = \frac{1}{\sqrt{2\pi^3}} \int d^3\tilde{r} e^{+i\tilde{p}\vec{r}} \tilde{\Psi}(\tilde{r}) = \mathcal{F}^{-1}\tilde{\Psi}$

$$\begin{aligned} \implies F(\vec{k}) &= \int d^3r \Psi^*(\vec{r})\Psi(\vec{r})e^{i\vec{k}\vec{r}} \\ &= \int d^3r \Psi^*(\vec{r})\frac{1}{\sqrt{2\pi^3}} \int d^3\tilde{p} e^{-i(\tilde{p}-\vec{k})\vec{r}} \tilde{\Psi}(\tilde{p}) \end{aligned}$$

with  $\vec{p} = \tilde{p} - \vec{k} \Rightarrow d^3p = d^3\tilde{p}$

$$\begin{aligned} \implies F(\vec{k}) &= \int d^3r \Psi^*(\vec{r})\frac{1}{\sqrt{2\pi^3}} \int d^3p e^{-i\vec{p}\vec{r}} \tilde{\Psi}(\vec{p} + \vec{k}) \\ &= \int d^3p \tilde{\Psi}(\vec{p} + \vec{k}) \left[ \frac{1}{\sqrt{2\pi^3}} \int d^3r e^{+i\vec{p}\vec{r}} \Psi(\vec{r}) \right]^* \\ &= \int d^3p \tilde{\Psi}(\vec{p} + \vec{k})\tilde{\Psi}^*(\vec{p}) = F(\vec{k}) \end{aligned}$$

$$\begin{aligned} \implies \rho(\vec{r}) &= \frac{1}{(2\pi)^3} \int d^3k F(\vec{k}) e^{-i\vec{k}\vec{r}} \\ &= \frac{1}{(2\pi)^3} \int d^3k \int d^3p \tilde{\Psi}(\vec{p} + \vec{k})\tilde{\Psi}^*(\vec{p}) e^{-i\vec{k}\vec{r}} \\ &= \frac{1}{(2\pi)^3} \int d^3k \int d^3p \tilde{\Psi}(\vec{p} + \vec{k}) e^{-i(\vec{p}+\vec{k})\vec{r}} \tilde{\Psi}^*(\vec{p}) e^{i\vec{p}\vec{r}} \end{aligned}$$

with  $\vec{p} + \vec{k} = \vec{p}' \Rightarrow d^3p' = d^3k$

$$\begin{aligned} \implies \rho(\vec{r}) &= \frac{1}{(2\pi)^3} \int d^3p' \int d^3p \tilde{\Psi}(\vec{p}') e^{-i\vec{p}'\vec{r}} \tilde{\Psi}^*(\vec{p}) e^{i\vec{p}\vec{r}} \\ &= \Psi^*(\vec{r})\Psi(\vec{r}) = \rho(\vec{r}) \end{aligned}$$

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$$\begin{aligned} \Rightarrow F(\vec{k}) &= \int d^3r \rho(\vec{r}) e^{i\vec{k}\vec{r}} \\ &= \int d^3r \rho(\vec{r}) \left( 1 + i\vec{k}\vec{r} - \frac{(\vec{k}\vec{r})^2}{2} + \dots \right) \end{aligned}$$

if  $\rho(\vec{r})$  spherically symmetric  $\Rightarrow \rho(\vec{r}) = \rho(r)$

$$\begin{aligned} \Rightarrow F(k^2) &= \iiint \left( 1 + ikr \cos(\vartheta) - \frac{k^2 r^2}{2} \cos^2(\vartheta) + \dots \right) \rho(r) r^2 dr d\cos(\vartheta) d\varphi \\ &= \iiint \rho(r) d^3r - \iiint \frac{k^2}{2} \cos^2(\vartheta) d\cos(\vartheta) \rho(r) r^4 dr d\varphi + \dots \end{aligned}$$

with  $\langle r^2 \rangle = \frac{\iiint r^2 \rho(r) d^3r}{\iiint \rho(r) d^3r}$  and  $\iiint \rho(r) d^3r = 1$

$$\Rightarrow F(k^2) = 1 - \frac{1}{6} k^2 \langle r^2 \rangle + \dots$$

$$\Rightarrow \langle r^2 \rangle = -6 \left( \frac{dF(k^2)}{dk^2} \right)_{k^2=0}$$