The electrostatic potential between two plates

Given two infinite parallel plates, held a fixed distance d apart at a constant potential difference, calculate the dependance of the potential difference between one plate and a point between them at, say, z.

In[1]:=
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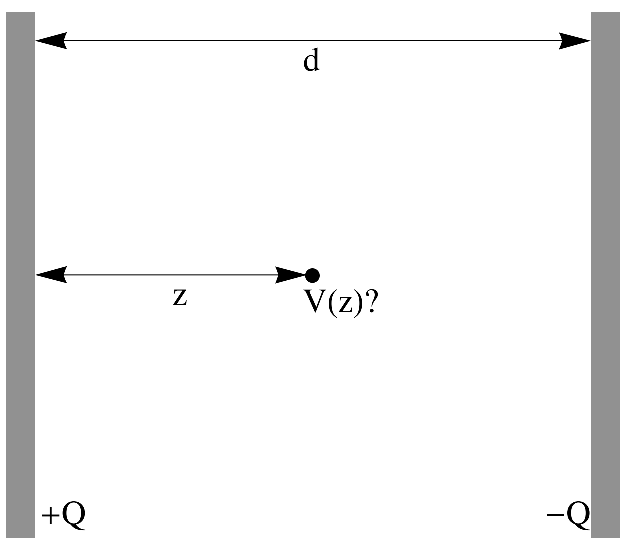
Graphics[{Gray, Rectangle[{0, 0}, {1, 18}],
  Rectangle[{20, 0}, {21, 18}], Black, Disk[{10.5, 9}, 0.25],
  Arrowheads[{-.05, .05}], Arrow[{{1, 17}, {20, 17}}],
  Arrowheads[{-.05, .05}], Arrow[{{1, 9}, {10.3, 9}}],
  Inset[Cell["d", FontSize → 16], {10.5, 16.25}],
  Inset[Cell["V(z)?", FontSize → 16], {11.5, 8}],
  Inset[Cell["+Q", FontSize → 16], {2, 0.75}],
  Inset[Cell["-Q", FontSize → 16], {19.25, 0.75}],
  Inset[Cell["z", FontSize → 16], {6, 8.25}]}]
```

Out[1]=



Solution from the potential of a charge distribution (the long way)

We start from Eq. 24-32 from page 638 giving the potential due to a continuous charge distribution

$$V[z\_] := \int \frac{1}{4\,\pi\epsilon_0}\,\frac{dq}{d[z]}$$

where dq is the infinitessimal element of charge and $d(x)$ is it's distance from the point x where we wish to measure the potential.

Our first order of business is going to be writing dq in a useful form. Since we're working with plates, we're going to need to integrate over a flat 2D area, so our charge density is going to be a surface charge, call it σ, and we'll need to pick an infinitessimal area. A simple one to use is a ring of radius r and thickness dr, giving

$$dq = 2\pi\sigma r\, dr$$

The next task is to turn $d(x)$ into something useful. Since this is supposed to be the distance between the charge element and the point we want to measure the potential of. It's convenient to take the origin of our coordinate system to be directly under the point whose potential we want to measure because then

$$d[z\_] := \sqrt{z^2 + r^2}$$

and we have

$$V[z\_] := \int \frac{1}{4\pi\epsilon_0} \frac{2\pi\sigma r\, dr}{\sqrt{z^2 + r^2}}$$

$$V[z\_] := \frac{\sigma}{2\epsilon_0} \int \frac{r\, dr}{\sqrt{z^2 + r^2}}$$

for an infinite plate r goes from zero to ∞

$$V[z\_] := \frac{\sigma}{2\epsilon_0} \int_0^\infty \frac{r}{\sqrt{z^2 + r^2}}\, dr$$

Now we ask *Mathematica* to integrate

In[2]:= $\int_0^\infty \frac{r}{\sqrt{z^2 + r^2}}\, dr$

Integrate::idiv : Integral of $\frac{r}{\sqrt{r^2 + z^2}}$ does not converge on {0, ∞}. ≫

Out[2]= $\int_0^\infty \frac{r}{\sqrt{r^2 + z^2}}\, dr$

Oops. Let's try it without explicit integration limits and see what's happening

In[3]:= $\int \frac{r}{\sqrt{r^2 + z^2}}\, dr$

Out[3]= $\sqrt{r^2 + z^2}$

Ah. What happens to this as $r \to \infty$? Kaboom.

First, do we believe *Mathematica*?

■ Doing the integral instead of asking for it

We start with

$$\int \frac{r}{\sqrt{r^2 + z^2}}\, dr$$

The first thing I note is that I've got a $\sqrt{r^2}$ term which is ... unpleasant. A good way to get rid of complicated things in integrals is to try substituion.

Let $u = r^2$ then $du = 2\, r\, dr$ and we've got

$$\frac{1}{2} \int \frac{2\, r}{\sqrt{r^2 + z^2}}\, dr = \frac{1}{2} \int \frac{1}{\sqrt{u + z^2}}\, du = \frac{1}{2} \int \left(u + z^2\right)^{-1/2} du$$

which is a little better but not much. Since substitution worked last time let $y = u + z^2$ then $dy = du$ and we have

$$\frac{1}{2} \int \left(u + z^2\right)^{-1/2} du = \frac{1}{2} \int y^{-1/2}\, dy$$

This I can do, giving

$$\frac{1}{2} \int y^{-1/2}\, dy = y^{1/2} = \left(u + z^2\right)^{1/2} = \left(r^2 + z^2\right)^{1/2} = \sqrt{r^2 + z^2}$$

■ Now that we believe the integral ...

In one sense we've got the answer to the integration as you can see what happens if we put in integration limits. Still, it's nice to get the computer to verify things even when they're pretty clear. So let's work around the infinity for the moment and try the integration with a dummy upper limit

In[4]:= $\text{Assuming}\left[\{x > 0, r > 0, z > 0, \text{Element}[\{x, z, r\}, \text{Reals}]\}, \int_0^x \frac{r}{\sqrt{z^2 + r^2}}\, dr\right]$

Out[4]= $-z + \sqrt{x^2 + z^2}$

Putting all the other stuff back in

$$V[z\_] := \frac{\sigma}{2\, \epsilon_0} \left(-z + \sqrt{x^2 + z^2}\right)$$

This is the answer for a finite disk of radius x but that's not what we want and it blows up for an infinite disk, which, if you think about it, makes sense. An infinite disk with a finite charge density contains an infinite amount of charge. It's not unreasonable, though I suppose not always necessary, that an infinite amount of charge produces an infinite potential.

So what do we do? Well, we're supposed to be working on two parallel plates and we've only figured out one of them, so let's work on the other.

Right off we're pretty close to done. The equation's going to have to be similar since the other plate is ... a plate too. But there are two differences. If the plates are a distance d apart and we're looking at a point that's z above one, it must be $d - z$ below the other and the charge on the second plate must be of oposite sign.

So we'll call our first potential V_b for bottom and the other V_t for top.

$$V_b[z\_] := \frac{\sigma}{2\,\epsilon_0}\left(-z + \sqrt{x^2 + z^2}\right)$$

$$V_t[z\_] := -\frac{\sigma}{2\,\epsilon_0}\left(-(d - z) + \sqrt{x^2 + (d - z)^2}\right)$$

The total potential is then the sum of the two

$$V[z\_] := V_b[z] + V_t[z]$$

$$V[z\_] := \frac{\sigma}{2\,\epsilon_0}\left(-z + \sqrt{x^2 + z^2}\right) - \frac{\sigma}{2\,\epsilon_0}\left(-(d - z) + \sqrt{x^2 + (d - z)^2}\right)$$

A little algebraic fiddling to make it pretty and move the irrelevent bits to the side (and going slow to make sure the minus signs behave)

$$V[z\_] := \frac{\sigma}{2\,\epsilon_0}\left(-z + \sqrt{x^2 + z^2} - \left(-(d - z) + \sqrt{x^2 + (d - z)^2}\right)\right)$$

$$V[z\_] := \frac{\sigma}{2\,\epsilon_0}\left(-z + \sqrt{x^2 + z^2} + (d - z) - \sqrt{x^2 + (d - z)^2}\right)$$

$$V[z\_] := \frac{\sigma}{2\,\epsilon_0}\left(-z + (d - z) + \sqrt{x^2 + z^2} - \sqrt{x^2 + (d - z)^2}\right)$$

$$V[z\_] := \frac{\sigma}{2\,\epsilon_0}\left(d - 2\,z + \sqrt{x^2 + z^2} - \sqrt{x^2 + (d - z)^2}\right)$$

This is the potential at a point *z* between two finite plates of radius *x* a distance *d* apart. How do we deal with the infinite part?

■ Going to ∞

Well, from a 'proof by eyeball' we can see what's going to happen when *x*, the radius of the plate, gets big: the two square root terms are going to cancel. However, it pays to try to prove it. You can't do science by hunches alone (though you can't do it without them either).

First, mechanically:

In[5]:= $$\text{Assuming}\left[\{x > 0, d > 0, z > 0, \text{Element}[\{x, d, z\}, \text{Reals}]\},\ \text{Limit}\left[d - 2\,z + \sqrt{x^2 + z^2} - \sqrt{x^2 + (d - z)^2},\ x \to \infty\right]\right]$$

Out[5]= $d - 2\,z$

That's good. We got the same answer as we did by eye. But how do you know the machine did it right? Perhaps more importantly, you're not going to be allowed to use *Mathematica* on any tests. So let's do the limit by hand.

The first thing I note is that the leading $d-2z$ doesn't matter. We're only interested in the parts that contain x. What is

$$\lim_{x\to\infty}\left(\sqrt{x^2+z^2}-\sqrt{x^2+(d-z)^2}\right)?$$

First we break it up

$$\lim_{x\to\infty}\sqrt{x^2+z^2}-\lim_{x\to\infty}\sqrt{x^2+(d-z)^2}$$

but this is about as far as we can go by just looking for simple rules for limits. Both of these go to ∞ giving us

$$\infty-\infty$$

which is undefined.

To see why, consider that $\lim_{x\to\infty}(x)-\lim_{x\to\infty}(x+1)$ gives the same $\infty-\infty$ result but the limit is clearly -1. The moral of the story is that $\infty-\infty$ can be anything, hence it's 'undefined'. Another example of the unfortunateness of $\infty-\infty$ more relevent to our situation is

$$\lim_{x\to\infty}\sqrt{x^2+z^2}-\lim_{x\to\infty}\sqrt{x^{314}+(d-z)^2}$$

Clearly here the limit should be $-\infty$.

The issue is that the limit of the difference of two terms can be effected by how fast the two terms grow. By eye we can see that in our case they grow at the same speed.

In general, however, this may not be obvious. One solution is to look at the limit of the ratio of the two terms. This technique will capture the problem of relative growth rates, since if one is growing much faster than the other the limit of the ratio will either go to zero or ∞.

For our case we have

$$\lim_{x\to\infty}\frac{\sqrt{x^2+z^2}}{\sqrt{x^2+(d-z)^2}}=\lim_{x\to\infty}\sqrt{\frac{x^2+z^2}{x^2+(d-z)^2}}=\sqrt{\lim_{x\to\infty}\left(\frac{x^2+z^2}{x^2+(d-z)^2}\right)}=\sqrt{\lim_{x\to\infty}\left(\frac{x^2}{x^2+(d-z)^2}+\frac{z^2}{x^2+(d-z)^2}\right)}$$

but now we're getting somewhere because

$$\sqrt{\lim_{x\to\infty}\left(\frac{x^2}{x^2+(d-z)^2}+\frac{z^2}{x^2+(d-z)^2}\right)}=\sqrt{\lim_{x\to\infty}\left(\frac{x^2}{x^2+(d-z)^2}\right)+\lim_{x\to\infty}\left(\frac{z^2}{x^2+(d-z)^2}\right)}=\sqrt{1+0}=1$$

which means the two terms are indeed equal in the limit of infinite radius. You are invited to consider the x^2 vs. x^{314} case above.

The 'answer'

So where were we? We were trying to figure out what this was for large x:

$$V[z\_] := \frac{\sigma}{2\,\epsilon_0}\left(d - 2\,z + \sqrt{x^2 + z^2} - \sqrt{x^2 + (d - z)^2}\right)$$

and the answer is now known to be

$$V[z\_] := \frac{\sigma}{2\,\epsilon_0}\,(d - 2\,z)$$

■ Potential vs. Potential Difference

But we're not really done. You can never *measure* 'the potential', you can only ever measure a *potential difference* (which Halliday & Resnick probably state in one sentence somewhere in the book and then seldom, if ever, mention again. Introductory physics textbooks have the problem of stuffing 400 years of understanding into 1,000 pages. The good ones invariably accomplish this by making every sentence count. This means you need to read the book carefully. Important things may well be said only one time.).

Ignoring the potential-difference-only bit for the moment, we can check our answer to see how well it does.

We know what the potential difference should be from one plate to the next, i.e. between $z = 0$ and $z = d$. Using our formula these give

$$V[0] := \frac{\sigma}{2\,\epsilon_0}\,d$$

$$V[d] := \frac{\sigma}{2\,\epsilon_0}\,(-d)$$

which means the potential difference between the two plates is

$$\Delta V := V[d] - V[0] = \frac{\sigma}{2\,\epsilon_0}\,(-d) - \frac{\sigma}{2\,\epsilon_0}\,d = -\frac{2\,\sigma\,d}{2\,\epsilon_0} = -\frac{\sigma\,d}{\epsilon_0}$$

Happily, this is the Official Answer (see eq. 25-8 and 25-7 on page 660, assuming I've got the same version of the book as you do.).

But we still need to figure out how exactly to write the potential at some distance z, since just plugging specific numbers into

$$V[z\_] := \frac{\sigma}{2\,\epsilon_0}\,(d - 2\,z)$$

gives answers like

$$V[0] := \frac{\sigma}{2\,\epsilon_0}\,d$$

when the answer you get experimentally is zero (i.e. when you're touching the negative bar the voltmeter says zero volts).

Well, we answered our problem a few lines above. What we want is not $V(z)$ but $\Delta V(z)$ since we can only ever measure potential *differences:*

$$\Delta V[z\_] := V[z] - V[0] = \frac{\sigma}{2\,\epsilon_0}(d - 2\,z) - \frac{\sigma}{2\,\epsilon_0}\,d = \frac{\sigma}{2\,\epsilon_0}(d - 2\,z - d) = -\frac{2\,\sigma\,z}{2\,\epsilon_0} = -\frac{\sigma\,z}{\epsilon_0}$$

$$\Delta V[z\_] := -\frac{\sigma\,z}{\epsilon_0}$$

which again is the Right Answer (though I don't think it actually appears in the textbook. It follows pretty directly, however, from the same section in chapter 25 mentioned above).

The careful student will note that there's a minus sign difference between what we have and what can be derived from the section in chapter 25. I leave you to ponder why that is and why it doesn't make what we did wrong.

Solution from Gauss' Law (the short way)

See section 23-8 for the field inside two parallel plates, it's only a few paragraphs of thinking plus a few lines of math. Add to this eqn 25-7 relating the potential and the field and you're done.

Gauss is your friend.

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DateString[]
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