Effective Operator for Double-Beta Decay

J. Engel

May 15, 2008

Fiorignone-Fest ’08
Usefulness of $\beta\beta$ Decay

Rate proportional to square of “effective neutrino mass”

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The Rate

Rate depends on nuclear stuff as well as $m_{\text{eff}}$:

$$T_{1/2}^{0\nu} = \sum_{\text{spins}} \int |Z_{\nu}|^2 \delta(E_{\nu 1} + E_{\nu 2} - Q_{\beta\beta}) \frac{d^3 p_1}{2\pi^3} \frac{d^3 p_2}{2\pi^3}$$
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Neglecting the induced-pseudoscalar term and momentum dependence in the weak nucleonic current, and summing over intermediate states in closure (a good approximation) gives

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with

$$M_{0\nu}^F = \langle f | \sum_{a,b} H(r_{ab}, E) \tau_a^+ | i \rangle, \quad M_{0\nu}^{GT} = \langle f | \sum_{a,b} H(r_{ab}, E) \vec{\sigma}_a \cdot \vec{\sigma}_b \tau_a^+ \tau_b^+ | i \rangle$$

$$H(r, E) \approx \frac{2R}{\pi r} \int_0^\infty dq \frac{\sin qr}{q + E - (E_i + E_f)/2}$$
Calculating the Matrix Elements

This is hard to calculate because

- Relevant nuclei are heavy (e.g. $^{136}\text{Xe} \rightarrow ^{136}\text{Te}$) and/or complicated (e.g. $^{76}\text{Ge} \rightarrow ^{76}\text{Se}$).
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- No measured neutrinoless decays with which to calibrate.
- $M_{fi}$ sensitive to delicate two-body space/spin correlations.
- Most of the operator’s strength is to excited states in the final nucleus.
Lots done since 1987

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<th>Neutrons</th>
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<th>Shell Model</th>
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**QRPA vs. Shell Model:**

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Neutrons

Large single-particle space; simple correlations within it.
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**QRPA vs. Shell Model:**

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- **QRPA** uses “G-matrix” interaction, adjusts strengths in particular channels to reproduce $2\nu$ decay, pairing gaps, single-beta decay and beta-strength functions.

- **Shell model** adjusts monopole part of microscopically derived interaction to fit binding energies, spectra.
Some QRPA Tuning...

\[
M^{2\nu}_{\nu} (\text{MeV}^{-1})
\]

\[
M^{0\nu}_{\nu}
\]

\[
g_{pp}
\]

- 9 levels
- 21 levels

Some QRPA Tuning...
Results can differ by factor of 2 or more

Shell-Model vs. QRPA Results
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But the decay operator should be adjusted alongside the Hamiltonian if the wave function is incomplete
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So far, adjustments are purely phenomenological:

- $g_A$ sometimes set to 1,
- short-range correlations included via a prescription.
### Effective Decay Operator for Shell Model

#### Bloch-Horowitz Equation

\[
H_{\text{eff}} = PHP + PHQ \frac{1}{E - QHP} QHP
\]

\[
P = \sum_{i \in \text{SM space}} |i\rangle\langle i| \quad Q = \sum_{\text{other } i} |i\rangle\langle i|
\]
Effective Decay Operator for Shell Model

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\[ H_{\text{eff}} = PHP + PHQ \frac{1}{E - QH} QHP, \quad |\Psi\rangle = \mathcal{N} \left[ P|\Psi\rangle + \frac{1}{E - QH} Q|\Psi\rangle \right] \]

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\[ P = \sum_{i \in \text{SM space}} |i\rangle\langle i| \quad \text{and} \quad Q = \sum_{\text{other } i} |i\rangle\langle i| \]

\[ H_{\text{eff}}(E_a) P |\Psi_a\rangle = E_a P |\Psi_a\rangle \quad \frac{\langle \Psi_a | P M_{\text{eff}}^P |\Psi_b\rangle}{\sqrt{\langle \Psi_a | P |\Psi_a\rangle \langle \Psi_b | P |\Psi_b\rangle}} = \langle \Psi_a | M |\Psi_b\rangle \]
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**Formulation as Rayleigh-Schrödinger perturbation theory**

- replaces \( E \) by unperturbed (single-particle) energy
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Formulation as Rayleigh-Schrödinger perturbation theory

- replaces \( E \) by unperturbed (single-particle) energy
- leads to diagrammatic series for energy and matrix elements of other operators
Define $G$ matrix by iterated sum over high-lying “two-particle” states:

$$G = V + \frac{V}{V} + \ldots$$
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Then expand in $G$ to get the effective interaction $H_{\text{eff}}$:

$$H_{\text{eff}}$$
By analogy, define a $\beta\beta$ operator that includes high-energy stuff — all ladders in $V$ with one insertion of $\mathcal{M}$:

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\mathcal{M}_{\text{high}} = \mathcal{M} + \mathcal{G} + \mathcal{M} + \mathcal{G} + \mathcal{M}
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Then expand in $G$ to get full effective operator $\mathcal{M}_{\text{eff}}$:
Note that you generate effective operator $\mathcal{M}_{\text{eff}}$ by replacing $G$ (\includegraphics[width=1cm]{G}) with $G + \epsilon\mathcal{M}$ (\includegraphics[width=1cm]{G_epsilon}) and working to first order in $\epsilon$.

$\mathcal{M}_{\text{high}}$ and many of the diagrams in which it enters can be obtained from Morten Hjorth-Jensen’s effective interaction code by using this trick on the corresponding diagrams for the effective Hamiltonian.
So far: diagrams in $0f_{5/2}, 1p, 0g_{9/2}$ model space with shell model transition densities for $^{82}\text{Se}$ from Poves et al.
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**Full Matrix Element (no forbidden terms)**

- Bare: 3.78
- With all first-order diagrams: 5.07
- Caurier et al.: 2.49*
- QRPA ($g_A = 1.25$): $\approx 6$

* strong short-range correlations
Effective Operator in Simple Pairing Model

\[ M_{0v} (\text{fm}^{-1}) \]

\[ \varepsilon = 10G \]

\[ \varepsilon = 20G \]
Conclusions

- Can carry this approach to higher order. Complete calculation at next order would be straightforward.
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- Ultimate answer will probably require nonperturbative treatment. But that’s not as far off as it used to be.