6.20) (a) Let's do a Taylor expansion instead

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} \; ; \; \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = \frac{2}{(1-x)^3} \; ; \; \frac{d^n}{dx^n}\left(\frac{1}{1-x}\right) = \frac{n!}{(1-x)^{n+1}} .$$

Evaluated at $x=0$ the n^{th} derivative is $n!$

Thus, $\frac{1}{1-x} = 1 + x + x^2 + \ldots$

Radius of convergence is 1.

(b) $Z = \sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} e^{-\beta(n+1/2)\hbar\omega}$.

Writing the energy interval $\hbar\omega$ as ϵ we get

$$Z = \sum_{n=0}^{\infty} e^{-\beta(n+1/2)\epsilon} = e^{-\beta\epsilon/2} \sum_{n=0}^{\infty} e^{-n\beta\epsilon}$$

$$= e^{-\beta\epsilon/2}\left\{1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon} + \ldots\right\}$$

$$= \frac{e^{-\beta\epsilon/2}}{1-e^{-\beta\epsilon}} = \frac{1}{e^{\beta\epsilon/2} - e^{-\beta\epsilon/2}}, \text{ since } e^{-\beta\epsilon} < 1 .$$

(c) $$\bar{E} = -\partial_\beta\left[\ln(Z)\right] = \partial_\beta \ln\left(e^{\beta\epsilon/2} - e^{-\beta\epsilon/2}\right) = \frac{(\epsilon/2)\left(e^{\beta\epsilon/2} + e^{-\beta\epsilon/2}\right)}{\left(e^{\beta\epsilon/2} - e^{-\beta\epsilon/2}\right)}$$

$$= \frac{\epsilon}{2}\coth\left(\frac{\beta\epsilon}{2}\right).$$ [Note: for small β, i.e., large T, we have $\coth(\beta\epsilon/2) \approx 2/\beta\epsilon$ and thus $\bar{E} \approx \beta^{-1} = kT$. This includes potential energy!]

(d) For N oscillators $U = N\bar{E} = \frac{N\epsilon}{2}\coth\left(\frac{\beta\epsilon}{2}\right)$.

(e) $C \equiv \left(\frac{\partial U}{\partial T}\right) = \frac{dU}{dT}$ in our case if N is fixed.

Thus, $$C = \frac{N\epsilon}{2}\cdot\frac{\epsilon}{2}\cdot\left\{-\operatorname{cosech}^2\left(\frac{\beta\epsilon}{2}\right)\right\}\left\{\frac{-1}{kT^2}\right\}$$

For small ϵ, i.e., $\beta\epsilon \ll 1$, we get $C \approx Nk$. [High T limit].

At low T, $\operatorname{cosech}^2\left(\frac{\beta\epsilon}{2}\right)$ as $\beta\to\infty$ will be $\frac{1}{\sinh^2\left(\frac{\beta\epsilon}{2}\right)}$

i.e., $\frac{16}{e^{2\beta\epsilon}}$, which goes to zero faster than $\frac{1}{kT^2}$ rises, so $C \to 0$ as $T \to 0$.