PHYS 503 Test 2

October, 2016

***Asterisked sections are for graduate students only.**

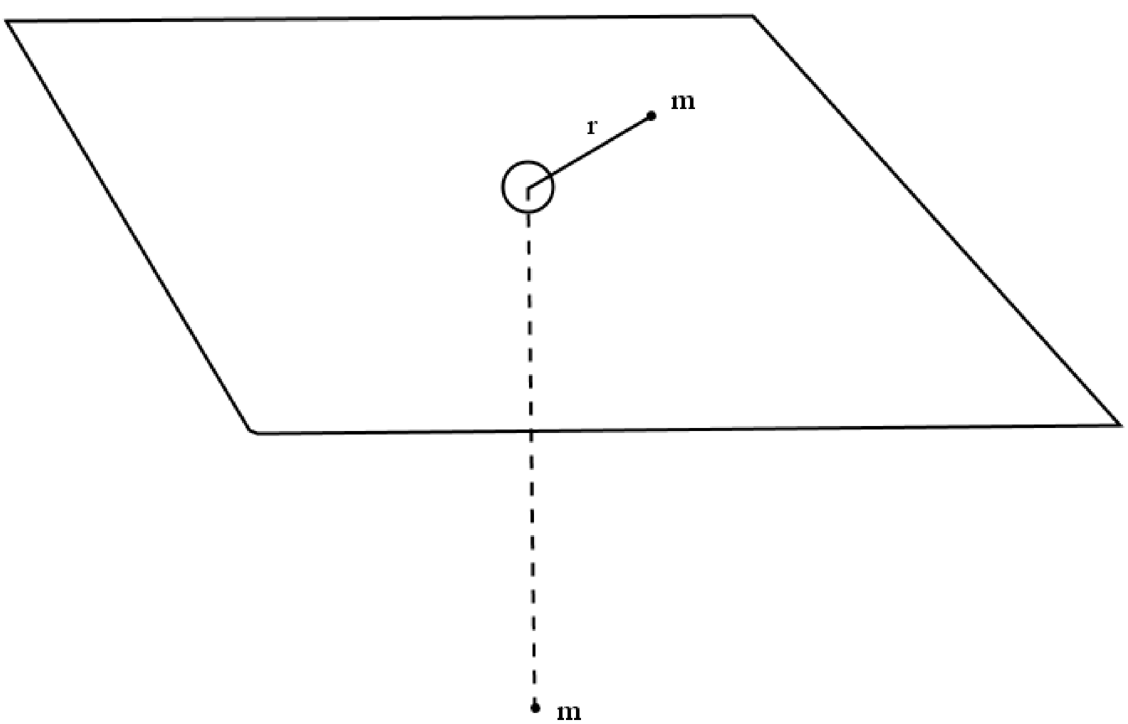
1. [10 points]

   Find the gravitational force on a mass m at a radius r inside a uniform spherical shell of mass M and with inner and outer radii R_1, and R_2, respectively.

2. [20 points]

   A string of length ℓ has a mass m at each end, and passes through a hole in a horizontal frictionless table. One mass moves horizontally on the table, while the other hangs vertically downwards; see the figure. Note that the hanging mass does **not** oscillate like a pendulum.

   (a) Write down the Lagrangian for this system.
   (b) Find the Euler-Lagrange equations of motion.
   (c) Find expressions for the generalized momenta.
   (d) Find the Hamiltonian for the system.
   *(e) Does H equal the total energy of the system? Explain.
   *(f) Are H and the generalized momenta constants of motion?



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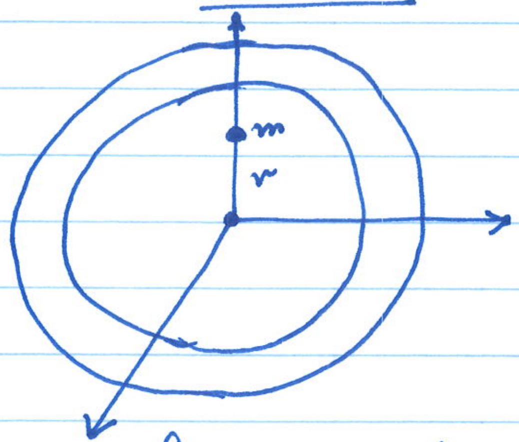
#1)



Without loss of generality, we use the symmetry to place m on the z-axis. Consider a ring of mass at a polar angle θ'. The potential due to this ring at m is given by

$$d\Phi(z=r) = G\int_{r'=R_1}^{r'=R_2} \frac{\rho\, 2\pi (r'^2 \sin\theta')\, d\theta'\, dr'}{\sqrt{r'^2\sin^2\theta' + (r - r'\cos\theta')^2}}$$

$$= G 2\pi\rho\, d(\cos\theta') \int_{r'=R_1}^{r'=R_2} dr' \frac{r'^2}{\sqrt{r'^2 + r^2 - 2rr'\cos\theta'}}$$

Integrating over θ' gives the potential:

$$\Phi(z=r) = G\, 2\pi\rho \int dr'\, r'^2 \int \frac{d(\cos\theta')}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}$$

$$= G\, 2\pi\rho \int r'^2 dr' \sqrt{r^2 + r'^2 - 2rr'\cos\theta} \left(\frac{2}{(-2rr')}\right)\Bigg|_{\cos\theta' = -1}^{\cos\theta' = R_2}$$

$$= 2\pi\rho G \int r'^2 dr' \left\{ (r - r') - (r + r') \right\} \left(-\frac{1}{rr'}\right)$$

$$= \frac{4\pi\rho G\,(R_2^2 - R_1^2)}{2} = \text{constant.}$$

$$\vec{F} = -m\vec{\nabla}\Phi \quad \Rightarrow \quad \vec{F} = 0.$$

#2) $T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{m}{2}\dot{r}^2 = m\dot{r}^2 + \frac{mr^2\dot{\theta}^2}{2}$

$U = mg\{-(\ell - r)\} = -mg(\ell - r)$

(a) Thus, $L = T - U = m\dot{r}^2 + \frac{mr^2\dot{\theta}^2}{2} + mg(\ell - r)$

(b) The Euler-Lagrange equations are

(i) $\underline{\underline{r}}$: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$ i.e.,

$$2m\ddot{r} = mr\dot{\theta}^2 - mg$$

(ii) $\underline{\underline{\theta}}$: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0$$ Let $mr^2\dot{\theta} = \ell$

$\Rightarrow \dot{\theta} = \frac{\ell}{mr^2}$.

i.e., $2m\dot{r}\dot{\theta} + mr^2\ddot{\theta} = 0$.

(c) $p_r = \frac{\partial L}{\partial \dot{r}} = 2m\dot{r} \Rightarrow \dot{r} = \frac{p_r}{2m}$

$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$

(d) $H = p_r\dot{r} + p_\theta\dot{\theta} - L = 2m\dot{r}^2 + mr^2\dot{\theta}^2 - L$

$= m\dot{r}^2 + \frac{mr^2\dot{\theta}^2}{2} - mg(\ell - r)$

$= m\left(\frac{p_r}{2m}\right)^2 + \frac{mr^2}{2}\left(\frac{p_\theta}{mr^2}\right)^2 - mg(\ell - r)$

$= \frac{p_r^2}{4m} + \frac{p_\theta^2}{2mr^2} - mg(\ell - r)$.

*(e) Yes, $H = E$ because the potential energy U does not depend on $\dot{r}$ or on $\dot{\theta}$, and the transformations from (x, y) to (r, θ) do not explicitly depend on time.

*(f) $\frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0$. Thus, H is a constant of motion.

We saw earlier, in part (b), that $\frac{d}{dt} p_\theta = 0$ and hence p_θ is a constant of motion.

However, $\frac{d}{dt} p_r \neq 0$ (also from part (b))

and hence p_r is <u>not</u> a constant of motion.