PHYS 503

9/9/16 Test 1 Solution

(1) (a) $\vec{\nabla}\times\vec{F} = (0,0,4)\,\frac{N}{m}$. No not a conservative force since $\vec{\nabla}\times\vec{F}\neq 0$.

(b) $W = \oint \vec{F}\cdot d\vec{r} = \int_A (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,da = \int 4\hat{z}\cdot\hat{z}\,da = 4\times 2.5\times 10^{18}\,\frac{N}{m}\times m^2$

$= 10^{19}$ J.

(2) (a) Let x increase downward and be zero on the ground.

Then $m\ddot{x} = mg - b\dot{x}$. Since $v = \dot{x}$ we have

$m\dot{v} = mg - bv$ or $\dot{v} = g - (b/m)v$

$\frac{dv}{g-(\frac{b}{m})v} = dt$ gives $\frac{\ln(g - \frac{bv}{m})}{(-b/m)} = t + \text{constant}.$

At $t=0$, $v=0$ and hence $\frac{\ln g}{(-b/m)} =$ the constant.

Thus, $\frac{\ln(g - bv/m)}{(-b/m)} - \frac{\ln(g)}{(-b/m)} = t$ (

$\ln\left(1 - \left(\frac{b}{mg}\right)v\right) = \left(\frac{-bt}{m}\right) \Rightarrow v = \left(\frac{mg}{b}\right)\left(1 - e^{-bt/m}\right)$

The equation of motion results from integrating this:

$x(t) = A + \int_0^t v(t')\,dt' = A + \left(\frac{mg}{b}\right)\left\{t + \frac{m}{b}e^{-bt/m}\right\}$

Adjust A to get $x(0) = -h$. $-h = A + \frac{m^2 g}{b^2}$

Thus, $x(t) = -\left(h + \frac{m^2 g}{b^2}\right) + \frac{mgt}{b} + \frac{m^2 g}{b^2}e^{-bt/m}$

$= -h + \frac{mgt}{b} - \frac{m^2 g}{b^2}\left(1 - e^{-bt/m}\right)$

(b) $x=0$ when $-h + \frac{mgT}{b} - \frac{m^2 g}{b^2}\left(1 - e^{-bT/m}\right) = 0$

At this time T,

$v_g = \frac{mg}{b}\left\{1 - e^{-bT/m}\right\}$

(c) $V_{terminal}$ is achieved as $t\to\infty$ and we get $\frac{mg}{b}$

#3) (a) $\omega_0 = \sqrt{\frac{k}{m}}$

(b) $\beta = \frac{b}{2m}$

$\omega_1 = \sqrt{\omega_0^2 - \beta^2}$

If $\omega_0 > \beta$ we have underdamped motion

$\omega_0 = \beta$ critically damped motion

$\omega_0 < \beta$ overdamped motion.

[Note: For small β $\omega_1 \approx \omega_0$.]

(c) $T = \frac{mv^2}{2} = \frac{m}{2}\left\{-\beta x - \omega_1 x_0 e^{-\beta t}\sin(\omega_1 t)\right\}^2$

$U = \frac{kx^2}{2} = \frac{k}{2}x^2$.

Thus, $T+U = \frac{kx^2}{2} + \frac{m}{2}\Big(\beta^2 x^2 + \omega_1^2 x_0^2 e^{-2\beta t}\sin^2\omega_1 t$
$+ 2\beta\omega_1 x x_0 e^{-\beta t}\sin(\omega_1 t)\Big)$

$\underset{(\text{for small }\beta)}{\cong} \frac{kx^2}{2} + \frac{kx_0^2}{2}e^{-2\beta t}\sin^2\omega_1 t + \frac{mx^2\beta^2}{2}$
$+ 2\beta\omega_1 x_0^2 e^{-2\beta t}\sin(\omega_1 t)\cos(\omega_1 t)$.

The last term averages to zero. The first 2 terms add to $\frac{kx_0^2}{2}e^{-2\beta t}$. Thus, neglecting $\frac{mx^2\beta^2}{2}$ which is quadratic in β we get

$E \approx E_0 e^{-2\beta t}$.