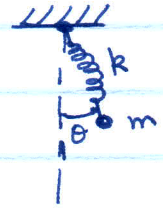
7-29)



$$\begin{cases} x = \ell \sin\theta \\ y = -\ell\cos\theta + \dfrac{at^2}{2} \end{cases} \qquad \begin{aligned} \dot{x} &= \dot{\ell}\sin\theta + \ell\cos\theta\,\dot{\theta} \\ \dot{y} &= -\dot{\ell}\cos\theta + \ell\sin\theta\,\dot{\theta} + at \end{aligned}$$

$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) \qquad U = mgy + \frac{k}{2}(\ell - b)^2$$

(a) Thus, $L = T - U$

$$= \frac{m}{2}\left\{\dot{\ell}^2 + \ell^2\dot{\theta}^2 + 2at\left(\ell\dot{\theta}\sin\theta - \dot{\ell}\cos\theta\right) + a^2t^2\right\} - mgy - \frac{k}{2}(\ell-b)^2$$

$$= \frac{m}{2}\left\{\dot{\ell}^2 + \ell^2\dot{\theta}^2 + 2at\left(\ell\dot{\theta}\sin\theta - \dot{\ell}\cos\theta\right) + a^2t^2\right\} + mg\left(\ell\cos\theta - \frac{at^2}{2}\right) - \frac{k}{2}(\ell-b)^2$$

The Euler-Lagrange equations for ℓ, θ $\quad \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}}\right) = \dfrac{\partial L}{\partial q}$ become:

ℓ: $$\frac{d}{dt}\left(m\dot{\ell} - mat\cos\theta\right) = m\ell\dot{\theta}^2 + mat\dot{\theta}\sin\theta + mg\cos\theta - k(\ell-b)$$

θ: $$\frac{d}{dt}\left(m\ell^2\dot{\theta} + mat\ell\sin\theta\right) = mat\ell\dot{\theta}\cos\theta + mat\dot{\ell}\sin\theta - mg\ell\sin\theta$$

(b) Due to the explicit time dependence in the transformation equations we need to be careful and not simply use $H = T + U$.

$$p_\ell = m\dot{\ell} - mat\cos\theta \Rightarrow \dot{\ell} = \frac{p_\ell}{m} + at\cos\theta \Rightarrow p_\ell\dot{\ell} = \frac{p_\ell^2}{m} + p_\ell at\cos\theta$$

$$p_\theta = m\ell^2\dot{\theta} + mat\ell\sin\theta \Rightarrow \dot{\theta} = \frac{p_\theta}{m\ell^2} - \frac{at}{\ell}\sin\theta \Rightarrow p_\theta\dot{\theta} = \frac{p_\theta^2}{m\ell^2} - p_\theta\frac{at}{\ell}\sin\theta$$

Now we can write

$$H = p_\ell\dot{\ell} + p_\theta\dot{\theta} - L = \frac{p_\ell^2}{m} + p_\ell at\cos\theta + \frac{p_\theta^2}{m\ell^2} - p_\theta\frac{at\sin\theta}{\ell} - L$$

7-29, p.2) $H = \frac{p_\ell^2}{m} + p_\ell \, at\cos\theta + \frac{p_\theta^2}{m\ell^2} - p_\theta \frac{at\sin\theta}{\ell}$

$$- \frac{m}{2}\left\{\left(\frac{p_\ell}{m} + at\cos\theta\right)^2 + \ell^2\left(\frac{p_\theta}{m\ell^2} - \frac{at\sin\theta}{\ell}\right)^2 + 2at\left(\ell\left(\frac{p_\theta}{m\ell^2} - \frac{at\sin\theta}{\ell}\right)\sin\theta - \left(\frac{p_\ell}{m} + at\cos\theta\right)\cos\theta\right) + a^2t^2\right\}$$

$$- mg\left(\ell\cos\theta - \frac{at^2}{2}\right) + \frac{k}{2}(\ell-b)^2$$

$$= \frac{p_\ell^2}{2m} - \frac{m}{2}a^2t^2\cos^2\theta + \frac{p_\theta^2}{2m\ell^2} - \frac{m}{2}a^2t^2\sin^2\theta$$

$$- \frac{at\,p_\theta}{\ell}\sin\theta + ma^2t^2\sin^2\theta + at\,p_\ell\cos\theta + ma^2t^2\cos^2\theta - \frac{m}{2}a^2t^2 - mg\left(\ell\cos\theta - \frac{at^2}{2}\right) + \frac{k}{2}(\ell-b)^2$$

$$= \frac{p_\ell^2}{2m} + \frac{p_\theta^2}{2m\ell^2} + at\,p_\ell\cos\theta - \frac{at}{\ell}p_\theta\sin\theta - mg\ell\cos\theta + \frac{mgat^2}{2} + \frac{k}{2}(\ell-b)^2$$

Now we can get equations for $\theta, \ell, p_\theta, p_\ell$:

$\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2} - \frac{at\sin\theta}{\ell}$ and $\dot\ell = \frac{\partial H}{\partial p_\ell} = \frac{p_\ell}{m} + at\cos\theta$

Similarly, using $\dot p_i = -\frac{\partial H}{\partial q_i}$ we get

$\dot p_\theta = at\,p_\ell\sin\theta + \frac{at\,p_\theta}{\ell}\cos\theta - mg\ell\sin\theta$ and

$\dot p_\ell = \frac{p_\theta^2}{m\ell^3} - \frac{at\,p_\theta\sin\theta}{\ell^2} + mg\cos\theta - k(\ell-b)$

(c) For small oscillations we can set $\cos\theta \approx 1$ and $\sin\theta \approx \theta$ to get from the Euler-Lagrange equations for L

ℓ-eqn: $\ddot\ell - a \cong g - \frac{k}{m}(\ell - b)$ [neglecting $\theta\dot\theta$, $\dot\theta^2$ terms]

i.e., $\ddot\ell + \left(\frac{k}{m}\right)\ell \cong (g+a) + \left(\frac{k}{m}\right)b$. Set $\omega = \sqrt{\frac{k}{m}}$ and this is the angular frequency of oscillations.

$T_\ell = 2\pi/\omega = 2\pi\sqrt{m/k}$.

Similarly, for θ we get $\ddot\theta + \frac{a}{\ell}\sin\theta \cong -\frac{g}{\ell}\sin\theta$

i.e., $\ddot\theta + \frac{(g+a)}{\ell}\theta \approx 0$. Thus, $T_\theta \approx 2\pi\sqrt{\ell/(g+a)}$.