5-11) $\vec{g} = -\frac{GM}{r^2}\hat{r} = \frac{1}{m}\vec{F} = -\frac{\vec{\nabla}U}{m}$ [For a mass m in the presence of M.]

$$\vec{\nabla}\cdot\vec{g} = \frac{-\nabla^2 U}{m} = -GM\,\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = -4\pi GM\,\delta^{(3)}(\vec{r}).$$

Thus, $\nabla^2 U = 4\pi GMm\,\delta^{(3)}(\vec{r})$ for M at the origin.

For a distributed mass

$$\nabla^2 U = 4\pi Gm\,\rho(\vec{r}).$$

The spherical surface and the volume it encloses contain no mass, and hence $\nabla^2 U = 0$ in that sphere. The average of U over the surface, or of $\Phi = \frac{U}{m}$ is thus

$$\langle\Phi\rangle_s = \frac{1}{4\pi R^2}\oint_S \Phi\,da.$$

On the surface $\Phi(\vec{r}) = \Phi(\vec{r}_c + \vec{R})$ where $\vec{r}_c$ = Location of center, $\vec{R}$ = vector from center

$$= \Phi(\vec{r}_c) + (\vec{R}\cdot\vec{\nabla})\,\Phi(\vec{r})\Big|_{\vec{r}=\vec{r}_c} + \frac{1}{2!}(\vec{R}\cdot\vec{\nabla})^2\,\Phi(\vec{r})\Big|_{\vec{r}=\vec{r}_c} + \cdots$$

Clearly, the average value of $(\vec{R}\cdot\vec{\nabla}\Phi)\big|_{\vec{r}=\vec{r}_c}$ over the surface is zero, since for every point on the surface there is another diametrically across with exactly the negative contribution. The same is true for the cross-terms in $(\vec{R}\cdot\vec{\nabla})^2$ but

$$R_x^2\left[\partial_x^2\Phi(\vec{r})\Big|_{\vec{r}=\vec{r}_c}\right] + R_y^2\left[\partial_y^2\Phi(\vec{r})\Big|_{\vec{r}=\vec{r}_c}\right] + R_z^2\left[\partial_z^2\Phi(\vec{r})\Big|_{\vec{r}=\vec{r}_c}\right]$$ survives.

Since the Laplacian $\nabla^2 U$ at the center vanishes, this is zero because $\langle R_x^2\rangle = \langle R_y^2\rangle = \langle R_z^2\rangle = \frac{R^2}{3}$.

Similarly, odd powers of $(\vec{R}\cdot\vec{\nabla})$ applied to Φ and evaluated at $\vec{r}=\vec{r}_c$ vanish when the $\vec{R}$-dependence is averaged over the sphere, and even powers vanish because $\nabla^2\Phi$ vanishes at the center. This proves that $\langle\Phi\rangle_s = \Phi_c$.