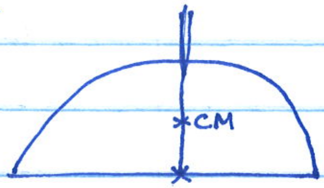
(a) Hemisphere



$$\rho = \frac{M}{\frac{2}{3}\pi R^3}$$

$$\vec{R}_{CM} = \frac{1}{M}\int \rho\, d^3r'\, \vec{r}\,' = \frac{1}{M}\int \frac{M}{\left(\frac{2}{3}\pi R^3\right)} \vec{r}\,' d^3r'$$

$$= \frac{3}{2\pi R^3}\int r'\hat{r}\, r'^2 dr'\, d(\cos\theta')\, d\phi'$$

$\theta' : 0 \to \frac{\pi}{2}$ only

$\cos\theta' : 0 \to 1$ only

Now try $\vec{r}\,' = (x', y', z')$

Noting that x', y' give zero ($d\phi'$ integrals) we get

$$\frac{3}{2\pi R^3}\int r'\cos\theta'\, r'^2\, dr'\, (2\pi)\, d(\cos\theta') \qquad (\text{for } z_{CM}).$$

$$= \frac{3}{2\pi R^3}\cdot 2\pi \cdot \frac{1}{2}\cdot\frac{R^4}{4} = \frac{3}{8}R.$$

(b) $$I_{ij} = \int d^3r'\,(r'^2\delta_{ij} - r_i' r_j')\,\rho$$

The off-diagonal elements are clearly zero.

$$I_{zz} = \int d^3r''(r''^2 - z''^2)$$

where $\vec{r}\,'$ is from the c.m.!

$$\vec{r}\,'' = \vec{r}\,' - \vec{R}_{CM}$$

Thus, $$I_{zz} = \int d^3r' \left\{(r'^2 + R_{CM}^2 - 2\vec{r}\,'\cdot\vec{R}_{CM}) - (z' - z_{CM})^2\right\}\rho$$

$$= \rho\int d^3r' \left\{(x'^2+y'^2) + R_{CM}^2 - 2z'z_{CM} + 2z'z_{CM} - z_{CM}^2\right\}$$

$$= \rho\int d^3r'\,(x'^2+y'^2) + (x_{CM}^2 + y_{CM}^2)\int d^3r'$$

$$= \rho\int d^3r'\,(R^2 - z'^2)$$

$$= 2\pi\cdot\rho\int r'^2\sin^2\theta\; r'^2 dr'\, d(\cos\theta')$$

$$= \frac{2\pi R^5}{5}\rho\left(1 - \frac{1}{3}\right) = \frac{2}{5}MR^2$$

(c) I_{xx} and I_{yy} are equal by symmetry.

$$I_{xx} = \int \rho \, d^3r' \, (y^2 + z^2)$$

For hemisphere center J_{xx} (see text eq. (11.49)), is given by

$$J_{xx} = \left(\frac{M}{\frac{2}{3}\pi R^3}\right) \int_0^1 d(\cos\theta') \int_0^{2\pi} d\phi' \int_0^R r'^2 dr' \, (y'^2 + z'^2)$$

$$= \frac{3}{2}\frac{M}{\pi R^3} \iiint r'^4 (\cos^2\theta' + \sin^2\theta' \sin^2\phi') \, dr' \, d\phi' \, d(\cos\theta')$$

$$= \frac{3M}{2\pi R^3} \int_0^R dr' \frac{r'^4}{3} (2\pi + 2\pi) = \frac{2M}{R^3} \cdot \frac{R^5}{5} = \frac{2}{5} MR^2 .$$

$$I_{xx} = J_{xx} - M\left(\left(\frac{3R}{8}\right)^2 - 0\right) = \left(\frac{2}{5} - \frac{9}{64}\right) MR^2 = \frac{83}{320} MR^2 .$$