Pion-nucleon scattering and the nucleon Σ term in an extended linear Σ model

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A modified linear Σ model that allows for $g_A = 1.26$ by addition of vector and pseudovector πN coupling terms was discussed by Bjorken and Nauenberg and by Lee. In this extended linear Σ model the elastic πN scattering amplitudes satisfy the relevant chiral low-energy theorems, such as the Weinberg-Tomozawa relation for the isovector πN scattering length and in some cases Adler's ''consistency condition.'' The agreement of the isospin symmetric πN scattering length with experiment is substantially improved in this extended Σ model as compared with the original linear one. We show that the nucleon sigma term (Σ_N) in the linear and the extended Σ models with three different kinds of chiral symmetry breaking terms are identical. Within the tree approximation the formal operator expression for the Σ_N term and the value extracted from the πN scattering matrix coincide. Large values of Σ_N are easily obtained without any $s\bar{s}$ content of the nucleon. Using chiral rotations the Lagrangian of this extended Σ model reproduces the lowest-order πN chiral perturbation theory Lagrangian.

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I. INTRODUCTION

Gell-Mann and Levy's (GML) linear Σ model is a principal example of spontaneously broken chiral symmetry in strong interactions [1]. It is known that the linear Σ model does not always give the correct phenomenology, e.g., the value of the isoscalar pion-nucleon scattering length is too large. We shall show that in the extended linear Σ model to be presented in this paper, the phenomenology is considerably improved compared to the original GML model. Another alleged drawback of the linear sigma model is that, apart from chiral symmetry, the model has not been connected directly to QCD. Recently, however, it has been shown that the model can be thought of as a low-energy effective theory of Coulomb gauge QCD, albeit in the unrealistic limit of maximal $\mathrm{U}_A(1)$ symmetry breaking [2].

Another ''weakness'' of the linear Σ model is that the value of the axial coupling strength g_A equals one. It is known that the one-loop ''radiative'' corrections in the linear Σ model lead to the renormalization of the nucleon part of the axial current [3], but it is not widely known how to incorporate that kind of correction, i.e., a value of $g_A \neq 1$, into an effective (tree-level) Lagrangian. In some publications a proposed ''solution'' is to multiply the total axial current $\mathbf{J}^a_{\mu 5} = \mathbf{A}^a_\mu + \mathbf{a}^a_\mu$ by g_A where the nucleon part of the axial current is $\mathbf{A}^a_\mu = \bar{\psi}\gamma_\mu\gamma_5(\boldsymbol{\tau}^a/2)\psi$, and the meson part of the axial current is $\mathbf{a}^a_\mu = \sigma\partial_\mu\boldsymbol{\pi}^a - \boldsymbol{\pi}^a\partial_\mu\sigma$. Another ''solution'' posits the same, but this time just for $\mathbf{A}^a_\mu$. Both of these ''solutions'' are inconsistent with the chiral symmetry of the model. The first one violates the chiral charge algebra by leading to

$$[Q^a_5, Q^b_5] = g_A^2 i\varepsilon^{abc}Q^c \neq i\varepsilon^{abc}Q^c. \tag{1}$$

The second ''solution'' leads to Eq. (1) for the nucleon part of the axial charge, and in addition to a nonconserved axial Noether current even in the chiral limit since the equations of motion have *not* been modified.

In an earlier publication [4] one of us reinitiated the study of a venerable, but little-known extension of the linear Σ model, see, e.g., Ref. [5]. This extension allows the nucleon axial coupling constant g_A to be different from unity without violating chiral symmetry. The extra term introduced in the linear Σ model is a nonrenormalizable, derivative-coupling term, analogous to the Pauli anomalous (electron) magnetic moment term that describes the finite one-loop radiative correction in QED, and that is often introduced into other effective Lagrangians. This extended linear Σ model allows one to study the g_A dependence of the πN scattering lengths $a_{\pi N}$ and of the nucleon Σ term Σ_N. It is well known that $a^{(-)}_{\pi N}$ depends crucially on the value of g_A, whereas the Σ_N dependence on g_A is unknown [6]. We shall display this dependence and show that a large value of Σ_N can easily be obtained without recourse to any $s\bar{s}$ component of the nucleon. We can also reproduce the new, tiny experimental value of the isoscalar πN scattering length $a^{(+)}_0$. Our methods and results are potentially important for studies of nuclear matter, because the quark condensate in nuclear matter is determined by the n-nucleon Σ terms [7,8], and the issue of (P-wave) pion condensation depends crucially on g_A being different from unity [9].

The purpose of this study is to use the extended linear Σ model to derive some of the low-energy theorems for the elastic πN scattering amplitude, to calculate the πN scattering lengths, and to discuss the nucleon Σ term Σ_N. We believe that at least some of the generally valid predictions of chiral symmetry are most economically obtained in this model. Throughout this paper we shall use the tree approximation, save for one illustrative example done at the one-loop self-consistent approximation level, shown in Appendix A. In order to explore the various possibilities, and to facilitate comparison with earlier studies of the Gell-Mann–Levy linear Σ model we introduce three different chiral symmetry breaking (χSB) terms, as in Refs. [6,9]. For two of the three χSB terms, the effects on the pion's mass appear at the tree level, whereas the third χSB term's effect is only visible at the one-loop level, see Appendix A.

This paper falls into six sections. In Sec. II we define the extended linear Σ model, present the χSB terms and the canonical field variables, and show that the Noether charges close the chiral algebra although $g_A \neq 1$. Section III is devoted to a derivation of the elastic πN scattering amplitude, the Adler consistency condition and the scattering lengths. In Sec. IV we examine the nucleon sigma term Σ_N, first from the (formal) operator point of view and second as extracted from the elastic πN scattering amplitude in the first Born approximation and draw conclusions from the comparison of the two methods. In Sec. V we examine the connection with the effective pion-nucleon chiral perturbation theory, and Sec. VI summarizes the results.

II. THE EXTENDED LINEAR Σ MODEL

The extended Σ model is the linear Σ model modified by adding a pseudovector pion-nucleon coupling to the pseudoscalar one [4]. This model allows a nucleon axial current with arbitrary $g_A(\neq 1)$. The Lagrangian density of this model is given by

$$\begin{aligned}\mathcal{L}=&\bar{\psi} i \partial\!\!\!/ \psi-g_0\bar{\psi}[\sigma+i\gamma_5\boldsymbol{\pi}\cdot\boldsymbol{\tau}]\psi+\frac{1}{2}[(\partial_\mu\sigma)^2+(\partial_\mu\boldsymbol{\pi})^2]\\&+\frac{1}{2}\mu_0^2(\sigma^2+\boldsymbol{\pi}^2)-\frac{\lambda_0}{4}(\sigma^2+\boldsymbol{\pi}^2)^2+\mathcal{L}_{\chi\mathrm{SB}}\\&+\left(\frac{g_A-1}{f_\pi^2}\right)\left[\left(\bar{\psi}\gamma_\mu\frac{\boldsymbol{\tau}}{2}\psi\right)\cdot(\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi})\right.\\&\left.+\left(\bar{\psi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\psi\right)\cdot(\sigma\partial^\mu\boldsymbol{\pi}-\boldsymbol{\pi}\partial^\mu\sigma)\right].\end{aligned} \tag{2}$$

We assume that the parameters λ_0 and μ_0^2 are positive, which ensures spontaneous symmetry breaking in the tree approximation. The last line in Eq. (2) is a nonrenormalizable derivative-coupling term, introduced by Bjorken and Nauenberg and by Lee [5]. We shall focus on some consequences of adding this term to the linear Σ model Lagrangian.

The chiral symmetry breaking (χSB) terms in the Lagrangian are those discussed in Refs. [6,9]

$$\mathcal{L}_{\chi\mathrm{SB}}=-\mathcal{H}_{\chi\mathrm{SB}}=\varepsilon_1\sigma-\varepsilon_2\boldsymbol{\pi}^2-\varepsilon_3\bar{\psi}\psi. \tag{3}$$

An example of a different χSB term is discussed in, e.g., Ref. [8]. Each one of the three terms in Eq. (3) separately breaks the chiral symmetry and is capable of shifting the pion mass, though not always in the tree approximation. Yet the three terms do not always predict the same physics in all specific instances. In particular they predict different shifts of the nucleon mass, see Ref. [6], and, e.g., we find a different Goldberger-Treiman (GT) relation: $g_A M=g_{\pi N}f_\pi+\varepsilon_3$.

As usual we choose the ground state of the model as the minimum of the meson interaction Lagrangian $\mathcal{L}_{\mathrm{meson}}^{\mathrm{int}}$ with respect to the $\sigma-$ and π fields. This means shifting the Σ field by its vacuum expectation value, $\langle\sigma\rangle_0\equiv f_\pi$, i.e., $\sigma=f_\pi+s$, where from the minimum requirement we obtain

$$(\mu_0^2-\lambda_0 f_\pi^2)f_\pi=-\varepsilon_1. \tag{4}$$

The meson interaction Lagrangian in the new field variable reads

$$\begin{aligned}-\mathcal{L}_{\mathrm{meson}}^{\mathrm{int}}=&\frac{1}{2}(m_\sigma^2 s^2+m_\pi^2\boldsymbol{\pi}^2)+\frac{1}{2f_\pi}(m_\sigma^2-m_\pi^2+2\varepsilon_2)\\&\times s(s^2+\boldsymbol{\pi}^2)+\frac{1}{8f_\pi^2}(m_\sigma^2-m_\pi^2+2\varepsilon_2)(s^2+\boldsymbol{\pi}^2)^2.\end{aligned} \tag{5}$$

The resulting nucleon, s-meson, and pion masses are

$$M=\varepsilon_3+g_0 f_\pi, \tag{6a}$$

$$m_\sigma^2=-\mu_0^2+3\lambda_0 f_\pi^2, \tag{6b}$$

$$m_\pi^2=-\mu_0^2+\lambda_0 f_\pi^2+2\varepsilon_2=\varepsilon_1/f_\pi+2\varepsilon_2. \tag{6c}$$

The axial-vector Noether current

$$\begin{aligned}\mathbf{J}_{\mu5}^a=&\left(\bar{\psi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\psi\right)^a-(\boldsymbol{\pi}\partial_\mu\sigma-\sigma\partial_\mu\boldsymbol{\pi})^a+\left(\frac{g_A-1}{f_\pi^2}\right)\\&\times\left[\left(\bar{\psi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\psi\cdot\boldsymbol{\pi}\right)\boldsymbol{\pi}^a+\sigma^2\left(\bar{\psi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\psi\right)^a\right.\\&\left.+\sigma\left(\bar{\psi}\gamma_\mu\frac{\boldsymbol{\tau}}{2}\psi\times\boldsymbol{\pi}\right)^a\right],\end{aligned} \tag{7}$$

is partially conserved in this model. The divergence of this axial current is

$$\partial^\mu\mathbf{J}_{\mu5}^a=(\varepsilon_1+2\varepsilon_2\sigma)\boldsymbol{\pi}^a-\varepsilon_3\bar{\psi}i\gamma_5\boldsymbol{\tau}^a\psi. \tag{8}$$

When we assume that the physical one-pion state, $|\boldsymbol{\pi}\rangle$, does not have any $|s\boldsymbol{\pi}\rangle$ or $|N\bar{N}\rangle$ components, the matrix element of the divergence of the axial current (using $\sigma=f_\pi+s$) for the one-pion-to-vacuum transition gives

$$m_\pi^2 f_\pi=\varepsilon_1+2f_\pi\varepsilon_2. \tag{9}$$

To see explicitly that the purely one-nucleon part of the axial current has acquired the coupling constant $g_A\neq 1$, Eq. (7) is rewritten with the shifted Σ field ($\sigma=f_\pi+s$), and we obtain

$$\begin{aligned}\mathbf{J}_{\mu5}^a=&g_A\left(\bar{\psi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\psi\right)^a+f_\pi\partial_\mu\boldsymbol{\pi}^a+(s\partial_\mu\boldsymbol{\pi}-\boldsymbol{\pi}\partial_\mu s)^a\\&+\left(\frac{g_A-1}{f_\pi^2}\right)\left[\left(\bar{\psi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\psi\cdot\boldsymbol{\pi}\right)\boldsymbol{\pi}^a+s(2f_\pi+s)\right.\\&\left.\times\left(\bar{\psi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\psi\right)^a+(f_\pi+s)\left(\bar{\psi}\gamma_\mu\frac{\boldsymbol{\tau}}{2}\psi\times\boldsymbol{\pi}\right)^a\right].\end{aligned} \tag{10}$$

The axial charge density, however,

$$\rho_5^a = \mathbf{J}_{05}^a = \psi^\dagger \gamma_5 \frac{\boldsymbol{\tau}^a}{2} \psi - (\boldsymbol{\pi}^a \Pi_\sigma - \sigma \boldsymbol{\Pi}_\pi^a), \tag{11}$$

retains its linear Σ model form when written in terms of canonical fields and their associated canonical momenta [10]:

$$\Pi_\sigma = \dot{\sigma} - \left(\frac{g_A - 1}{f_\pi^2}\right)\left(\psi^\dagger \gamma_5 \frac{\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{2} \psi\right), \tag{12a}$$

$$\Pi_\pi^a = \dot{\boldsymbol{\pi}}^a + \left(\frac{g_A - 1}{f_\pi^2}\right)\left[\left(\psi^\dagger \frac{\boldsymbol{\tau}\times\boldsymbol{\pi}}{2} \psi\right)^a + \sigma \psi^\dagger \gamma_5 \frac{\boldsymbol{\tau}^a}{2} \psi\right]. \tag{12b}$$

The axial charge density, Eq. (11), and the vector charge density

$$\rho^a = \mathbf{J}_0^a = \psi^\dagger \frac{\boldsymbol{\tau}^a}{2} \psi + \varepsilon^{abc}(\boldsymbol{\pi}^b \boldsymbol{\Pi}_\pi^c), \tag{13}$$

close the algebra

$$[\rho^a(0,\mathbf{x}), \rho^b(0,\mathbf{y})] = i\varepsilon^{abc}\rho^c(0,\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y}), \tag{14a}$$

$$[\rho_5^a(0,\mathbf{x}), \rho_5^b(0,\mathbf{y})] = i\varepsilon^{abc}\rho^c(0,\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y}), \tag{14b}$$

$$[\rho_5^a(0,\mathbf{x}), \rho^b(0,\mathbf{y})] = i\varepsilon^{abc}\rho_5^c(0,\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y}), \tag{14c}$$

when we assume the canonical (anti)commutation relations

$$\{\psi^a(0,\mathbf{x}), \Pi_{\boldsymbol{\psi}}^b(0,\mathbf{y})\} = i\,\delta^{ab}\,\delta(\mathbf{x}-\mathbf{y}), \tag{15a}$$

$$[\sigma(0,\mathbf{x}), \Pi_\sigma(0,\mathbf{y})] = i\,\delta(\mathbf{x}-\mathbf{y}), \tag{15b}$$

$$[\pi^a(0,\mathbf{x}), \Pi_{\boldsymbol{\pi}}^b(0,\mathbf{y})] = i\,\delta^{ab}\,\delta(\mathbf{x}-\mathbf{y}). \tag{15c}$$

Thus we see that in this extended Σ model only the spatial part of the nucleon axial current is renormalized and the algebra of the charge operators is satisfied.

III. THE ELASTIC πN SCATTERING AMPLITUDE

We follow the discussion and methods of the linear Σ model in Ref. [6], but extended to include the new terms in the Lagrangian shown in the last line of Eq. (2). The main consequence of this modified Lagrangian is that the original πN coupling constant g_0 is renormalized to $g_{\pi N} = g_0[1 + (g_A - 1)(M/g_0 f_\pi)]$, where the nucleon mass is $M = g_0 f_\pi + \varepsilon_3$. This leads to a different set of S-wave scattering lengths and to the GT relation written above. Otherwise in the tree approximation the nucleon Σ terms are identical to those found by Campbell [6] as we show below.

A. The scattering amplitude

The elastic πN scattering amplitude T is usually written in terms of its two isospin and two Dirac matrix components as follows:

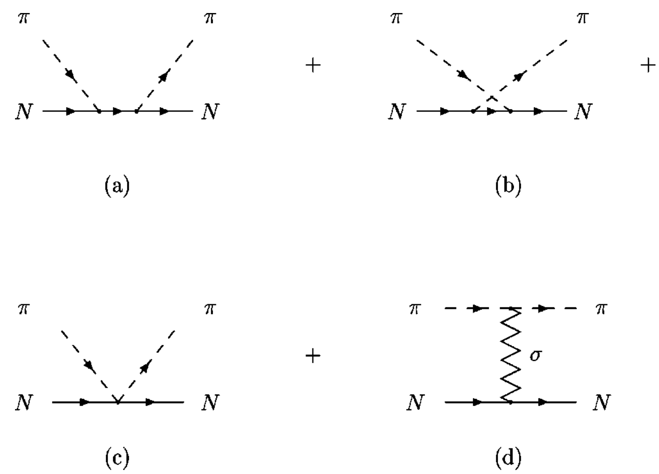

FIG. 1. The elastic πN scattering amplitude: (a) the direct and (b) crossed nucleon-pole diagrams, (c) the contact, or sea-gull diagram, and (d) the σ-meson-pole diagram. The dashed line denotes a pion; the zig-zag line denotes a σ meson, the solid line denotes a nucleon.

$$T_{\alpha\beta} = T^{(+)}\delta_{\alpha\beta} + T^{(-)}\frac{1}{2}[\boldsymbol{\tau}_\alpha, \boldsymbol{\tau}_\beta],$$

$$T = A + B\frac{1}{2}(\not{k}_1 + \not{k}_2), \tag{16}$$

where the incoming and outgoing pion's momenta are k_1 and k_2, and α and β are their isospin indices. An explicit calculation of the four tree-level diagrams in Fig. 1 leads to

$$A^{(+)} = \left(\frac{g_0}{f_\pi}\right)\left[\left(\frac{m_\sigma^2 - m_\pi^2 + 2\varepsilon_2}{m_\sigma^2 - t}\right) + 2(g_A - 1) + (g_A - 1)^2\left(\frac{M}{g_0 f_\pi}\right)\right], \tag{17a}$$

$$A^{(-)} = 0, \tag{17b}$$

$$B^{(+)} = g_0^2\left[1 + (g_A - 1)\left(\frac{M}{g_0 f_\pi}\right)\right]^2\left[\frac{1}{M^2 - s} - \frac{1}{M^2 - u}\right], \tag{17c}$$

$$B^{(-)} = g_0^2\left[1 + (g_A - 1)\left(\frac{M}{g_0 f_\pi}\right)\right]^2\left[\frac{1}{M^2 - s} + \frac{1}{M^2 - u}\right] - \frac{1}{2f_\pi^2}(g_A^2 - 1), \tag{17d}$$

where s, t and u are the standard Mandelstam variables, and $s + t + u = 2M^2 + k_1^2 + k_2^2$. Below we will use the traditional kinematical variables in the expressions for the amplitudes:

$$\nu = \frac{1}{4M}(k_1 + k_2)\cdot(p_1 + p_2) = \frac{s - u}{4M}, \tag{18a}$$

$$\nu_B = -\frac{k_1 \cdot k_2}{2M} = \frac{t - k_1^2 - k_2^2}{4M}. \tag{18b}$$

We follow standard notation and use $D^{(\pm)}$ as an abbreviation for

$$D^{(\pm)} \equiv A^{(\pm)} + \nu B^{(\pm)}. \tag{19}$$

In the tree approximation the extended Σ model isospin antisymmetric amplitude $D^{(-)}$ then reads for on- and off-mass-shell pions

$$D^{(-)}(\nu, \nu_B, k_1^2, k_2^2) = \left(\frac{g_{\pi N}^2}{M}\right)\nu\left[\frac{\nu_B}{\nu_B^2 - \nu^2} - \left(\frac{1 - g_A^{-2}}{2M}\right)\right.$$
$$\left.\times\left(1 + g_A^{-1}\frac{2\varepsilon_3}{M}\right)\right] + \mathcal{O}(\nu\varepsilon_i^2), \tag{20}$$

where we have used our GT relation. The second term in the square bracket $\propto (1 - g_A^{-2})/2M$ is absent in the regular Σ model where $g_A = 1$. To obtain the tree-level isospin symmetric amplitude we rewrite Eq. (17a) as follows:

$$A^{(+)} = \left(\frac{g_{\pi N}^2}{M}\right)\left[1 - g_A^{-2}\left(\frac{m_\pi^2 - t - 2\varepsilon_2}{m_\sigma^2 - t}\right) + g_A^{-2}\frac{\varepsilon_3}{M}\right] + \mathcal{O}(\varepsilon^2), \tag{21}$$

and Eq. (17c) is rewritten as

$$B^{(+)} = \frac{g_{\pi N}^2}{M}\frac{\nu}{\nu_B^2 - \nu^2} + \mathcal{O}(\varepsilon^2), \tag{22}$$

which gives, even for off-mass-shell pions,

$$D^{(+)}(\nu, \nu_B, k_1^2, k_2^2) = \left(\frac{g_{\pi N}^2}{M}\right)$$
$$\times\left\{\frac{\nu_B^2}{\nu_B^2 - \nu^2} - g_A^{-2}\left(\frac{m_\pi^2 - t - 2\varepsilon_2}{m_\sigma^2 - t}\right)\right.$$
$$\left. + g_A^{-2}\frac{\varepsilon_3}{M}\right\} + \mathcal{O}(\varepsilon^2). \tag{23}$$

Note that Eq. (23) is zero for $\nu = \nu_B = 0$ and $t = m_\pi^2$ only if $\varepsilon_2 = \varepsilon_3 = 0$. This means both the isospin symmetric $D^{(+)}$ and antisymmetric $D^{(-)}$ amplitudes have Adler zeros *only if PCAC, in its narrow definition, is satisfied as an operator equation* in the extended Σ model. This can also be seen by following Campbell's analysis [6] of the original linear Σ model. The main difference from the original linear Σ model is that the πN coupling constant is renormalized from g_0 in the original linear Σ model to $g_{\pi N} = g_A g_0[1 + (\varepsilon_3/g_0 f_\pi)(1 - g_A^{-1})]$, see Eq. (17c), and that the GT relation becomes $g_A M = g_{\pi N} f_\pi + \varepsilon_3$, after ''turning on'' $\varepsilon_3 \neq 0$, i.e., at the tree-level the GT relation acquires an ''anomaly'' $\propto \varepsilon_3$.

B. Scattering lengths

The πN scattering lengths are given by Eqs. (17a)–(17d):

$$a_0^{(\pm)} = \frac{D_{\text{threshold}}^{(\pm)}}{4\pi(1 + m_\pi/M)} = \frac{1}{4\pi(1 + m_\pi/M)}$$
$$\times[A^{(\pm)} + m_\pi B^{(\pm)}]_{\text{threshold}}, \tag{24}$$

which leads to the standard result for the isospin-symmetric scattering length in the $\varepsilon_2 = \varepsilon_3 = 0$ (but $\varepsilon_1 \neq 0$ since $m_\pi \neq 0$) limit

$$a_0^{(+)} = \frac{g_{\pi N}^2}{4\pi(1 + m_\pi/M)}\left(\frac{m_\pi}{M}\right)\left[1 - \frac{1}{1 - (m_\pi/2M)^2}\right.$$
$$\left. - g_A^{-2}\left(\frac{m_\pi}{m_\sigma}\right)^2\right]\frac{1}{m_\pi}$$
$$\simeq \frac{g_{\pi N}^2}{4\pi(1 + m_\pi/M)}\left(\frac{m_\pi}{M}\right)$$
$$\times\left[-\left(\frac{m_\pi}{2M}\right)^2 - g_A^{-2}\left(\frac{m_\pi}{m_\sigma}\right)^2\right]\frac{1}{m_\pi}. \tag{25}$$

The value for $a_0^{(+)}$ is smaller than the value in the ordinary linear Σ model, see, e.g., Delorme *et al.* [8], due to the factor g_A^{-2} in front of the second term. This will be discussed further in Sec. IV C.

The isospin antisymmetric scattering length equals the standard Weinberg-Tomozawa result

$$a_0^{(-)} = \frac{g_{\pi N}^2}{4\pi(1 + m_\pi/M)}\left(\frac{m_\pi}{M}\right)^2\left[\frac{1}{1 - (m_\pi/2M)^2} - (1 - g_A^{-2})\right]$$
$$\times\left(\frac{1}{2m_\pi}\right) + \mathcal{O}(\varepsilon_3^2) \simeq \frac{g_{\pi N}^2}{8\pi(1 + m_\pi/M)}\left(\frac{m_\pi}{M}\right)^2\left(\frac{1}{g_A^2 m_\pi}\right)$$
$$+ \cdots. \tag{26}$$

In the case when $\varepsilon_i \neq 0$, $i = 1,2,3$, we have

$$a_0^{(+)} \simeq \frac{-g_{\pi N}^2}{4\pi(1 + m_\pi/M)}\left(\frac{m_\pi}{M}\right)\left\{\left(\frac{m_\pi}{2M}\right)^2\right.$$
$$\left. + g_A^{-2}\left[\left(\frac{m_\pi^2 - 2\varepsilon_2}{m_\sigma^2}\right) - \left(\frac{\varepsilon_3}{M}\right)\right]\right\}\frac{1}{m_\pi} + \mathcal{O}(\varepsilon_i^2), \tag{27}$$

for the isoscalar scattering length. Note the negative sign in front of the ε_3 term which allows for either sign of this scattering length. The isovector scattering length is

$$a_0^{(-)} \simeq \frac{g_{\pi N}^2}{8\pi(1 + m_\pi/M)}\left(\frac{m_\pi}{M}\right)^2\frac{1}{g_A^2}\frac{1}{m_\pi} + \mathcal{O}(\varepsilon_3^2), \tag{28}$$

unchanged from the Weinberg-Tomozawa result. To compare these results with experiment we will determine the values of the χSB coefficients ε_i from some other source, see Sec. IV C and Appendix B. However, as all three χSB terms in Eq. (3) with their full strengths are not possible without overcounting, some care with the interpretation of these results is necessary.

IV. THE Σ TERM

The pion-nucleon Σ_N term is of importance for investigations of the $\langle\bar{\psi}\psi\rangle$ condensate in nuclear matter [7,8], and in determination of the flavor content of the nucleon [11]. As we shall show the extended Σ model gives a very interesting answer to the question of the flavor content of the nucleon. First we discuss the nucleon Σ_N term as obtained from the Σ operator and then evaluate the nucleon Σ_N term from the πN amplitude. We shall show that the connected one-nucleon matrix element of the Σ term operator coincides with another (operational) definition of the nucleon Σ term based on the pion-nucleon elastic scattering amplitude in the tree approximation. Finally we make a short estimate of the possible values of the Σ_N term in this model and also discuss the possible values of the πN scattering lengths.

A. Operator definition

The Σ operator is defined as

$$\Sigma^{ab}=[Q_5^a,[Q_5^b,\mathcal{H}_{\chi\mathrm{SB}}]],$$

$$\Sigma=\frac{1}{3}\sum_{a=b=1}^{3}\Sigma^{ab}. \tag{29}$$

Using a chiral Ward identity this operator appears after two applications of Sakurai's "master formula" to any elastic S-matrix element with one pion in the initial and one in the final state [12,13]. Here a,b are the flavor indices of the axial charge $Q_5^a=\int d\mathbf{x}\rho_5^a$ appropriate to the corresponding pseudoscalar mesons (pions), and $\mathcal{H}_{\chi\mathrm{SB}}$ is the chiral symmetry-breaking Hamiltonian density. In principle all of the objects entering Eq. (29) are meant to be exact Heisenberg representation operators. As we do not have exact solutions to the quantum-field equations of motion, we will discuss two approximate matrix elements of the Σ^{ab} operator for two cases: (i) the vacuum expectation value $\langle 0|\Sigma|0\rangle$ and (ii) the nucleon expectation value of its volume integral $\langle N|\int d\mathbf{x}\Sigma(x)|N\rangle$. The vacuum matrix element is well understood [14], so it leads to valuable constraints on the form of the χSB terms. As for the nucleon matrix element, we compare the results obtained from the above operator definition using the equations-of-motion, with another derivation based on the off-shell elastic πN scattering amplitude.

1. The Σ vacuum expectation value

The vacuum expectation value of the Σ operator yields Dashen's formula [13]

$$(fm^2f)^{ab}=f_a m_{ab}^2 f_b=-\langle 0|[Q_5^a,[Q_5^b,\mathcal{H}_{\chi\mathrm{SB}}]]|0\rangle. \tag{30}$$

This formula describes the lowest order χSB correction to the otherwise vanishing pseudoscalar meson mass squared (m_{ps}^2) for arbitrary chiral symmetry-breaking terms in the Hamiltonian density $\mathcal{H}_{\chi\mathrm{SB}}$. When the χSB term is taken to be the current quark mass in the QCD Hamiltonian

$$\mathcal{H}_{q\chi\mathrm{SB}}=\bar{q}m_q^0 q=m_u^0\bar{u}u+m_d^0\bar{d}d,$$

Eq. (30) yields

$$\begin{aligned}(f_{\mathrm{ps}}m_{\mathrm{ps}}^2 f_{\mathrm{ps}})^{ab}&=-\langle 0|[Q_5^a,[Q_5^b,\bar{q}m_q^0 q]]|0\rangle\\ &=-\langle 0|\bar{q}\left\{\left\{m_q^0,\frac{\lambda^a}{2}\right\},\frac{\lambda^b}{2}\right\}q|0\rangle,\end{aligned} \tag{31}$$

where λ^a, are the Gell-Mann matrices. By averaging over $a=1,2,3$ one finds the Gell-Mann–Oakes–Renner (GMOR) relation between the pion mass and decay constant on one hand and the current quark mass Hamiltonian vacuum expectation value on the other:

$$m_\pi^2 f_\pi^2=-[m_u^0\langle 0|\bar{u}u|0\rangle+m_d^0\langle 0|\bar{d}d|0\rangle], \tag{32}$$

To make contact with our previous discussion we apply Eq. (30) to our extended Σ model with the three kinds of χSB terms of Eq. (3). We use the canonical commutation relations, Eqs. (15a),(15b), and the axial charge, Eq. (11), to obtain

$$\begin{aligned}&[Q_5^a,[Q_5^b,\mathcal{H}_{\chi\mathrm{SB}}(0)]]\\ &\quad=-\varepsilon_1\sigma\delta^{ab}-2\varepsilon_2(\sigma^2\delta^{ab}-\boldsymbol{\pi}^a\boldsymbol{\pi}^b)+\varepsilon_3\bar{\psi}\psi\delta^{ab}.\end{aligned} \tag{33}$$

Taking the vacuum expectation value of this expression we find

$$(m_\pi f_\pi)^2=\varepsilon_1\langle 0|\sigma|0\rangle+2\varepsilon_2\langle 0|\sigma^2|0\rangle-\varepsilon_3\langle 0|\bar{\psi}\psi|0\rangle, \tag{34}$$

This relation goes beyond the tree approximation of Eq. (6c) as we show in Appendix A. We shall first examine Eq. (34) for the three distinct types of the χSB Hamiltonian in order to determine/normalize the values of the coefficients ε_i.

(i) $\varepsilon_i=0$ for $i=2$ and 3 leads to

$$\varepsilon_1\langle 0|\sigma|0\rangle=(m_\pi f_\pi)^2, \tag{35}$$

i.e., $\varepsilon_1=m_\pi^2 f_\pi$.

(ii) $\varepsilon_i=0$ for $i=1$ and 3 leads to

$$2\varepsilon_2\langle 0|\sigma^2|0\rangle=(m_\pi f_\pi)^2, \tag{36}$$

i.e., $\varepsilon_2=\frac{1}{2}m_\pi^2$.

(iii) $\varepsilon_i=0$ for $i=1$ and 2 leads to the relation

$$-\varepsilon_3\langle 0|\bar{\psi}\psi|0\rangle=(m_\pi f_\pi)^2. \tag{37}$$

We remark that this last relation looks similar to a nucleonic version of the GMOR relations, Eq. (32). To make this analogy more obvious, we introduce the explicit χSB "bare" nucleon mass matrix in our extended Σ model Lagrangian, Eq. (2), and compare it with $\mathcal{L}_{\chi\mathrm{SB}}$, Eq. (3). The corresponding χSB Hamiltonian density

$$\mathcal{H}_{N\chi\mathrm{SB}}=\bar{\psi}M_N^0\psi=M_p^0\bar{p}p+M_n^0\bar{n}n,$$

is used in Eq. (30) to obtain the relation

$$m_\pi^2 f_\pi^2=-[M_p^0\langle\bar{p}p\rangle_0+M_n^0\langle\bar{n}n\rangle_0]. \tag{38}$$

The obvious conclusion is that $\varepsilon_3=M_N^0$, the averaged ''bare'' nucleon mass, as expected from Eqs. (3) or (6a). We naturally express in terms of the current quark masses $M_N^0=3\overline{m}_q^0=\frac{3}{2}(m_u^0+m_d^0)\simeq 23$ MeV.

The basic underlying assumption of chiral perturbation theory as an effective hadronic field theory of QCD is that the χSB part of the Hamiltonian is a small perturbation. Two theories with different degrees of freedom (DF), e.g., quarks in one and hadrons in another, can be viewed as effectively mirroring each other provided both satisfy the same chiral symmetry transformations. For example, in a model with hadronic DF[1] the χSB part due to the current quark mass term in QCD is effectively mirrored in a pion mass term (plus possibly other terms with the same transformation properties). Chiral perturbation theory goes one step further and includes (to a given chiral order) all possible χSB terms in the Hamiltonian. The so-called low energy coefficients multiplying these χSB terms are then fit to the experimental data, though they could also be modelled in an underlying quark model [15].

In the following we argue that the cases (i), (ii), and (iii) could be interchangeable, at least as far as the nonzero pion mass is concerned. We wish to establish to what extent this interchangeability of the χSB terms actually holds in various approximations. (They certainly are not equivalent when it comes to nonvacuum matrix elements of the Σ term, as we shall show below.) In the tree approximation the first two terms on the right-hand side of Eq. (34) are the same as those in Eq. (9). Thus we see that the bare (current) nucleon mass term with $\varepsilon_3\neq 0$ does not lead to a massive pion in the tree approximation. In Appendix A we show how the bare nucleon mass $\overline{M}_N^0=\varepsilon_3\neq 0$ produces a nonzero pion mass $m_\pi\neq 0$ in agreement with the Dashen formula, Eq. (34), at the one-nucleon-loop self-consistent approximation level. One immediate conclusion is that the nominally identical forms of χSB terms in Eq. (34) do not always produce the same kinds of effects at the same level of approximation, even if the approximations conserve chiral-symmetry in the chiral limit $\varepsilon_i=0$, $i=1,2,3$.

Another consequence of Eq. (34) is that if one assumes the existence of more than one χSB term, then not all of such terms can have their ''full'' strengths. Specifically, if one wishes to have more than one χSB term in the Hamiltonian, Eq. (3), the coefficients ε_i must be rescaled. The new ''scaling coefficients'' α_i are defined as

$$\varepsilon_1=\alpha_1 m_\pi^2 f_\pi, \tag{39a}$$

$$\varepsilon_2=\alpha_2\frac{1}{2}m_\pi^2, \tag{39b}$$

$$\varepsilon_3=\alpha_3\overline{M}_N^0, \tag{39c}$$

subject to the condition of Eq. (34) that $\Sigma_{i=1}^3\alpha_i=1$. Similar problems arise in other quantities sensitive to χSB terms, such as the scattering lengths, Eqs. (27) and (28) as in, e.g., Ref. [8].

2. The nucleon Σ *term*

The nucleon Σ term (Σ_N) is, by definition, the connected elastic one-nucleon matrix element of the spatial (volume) integral[2] of the Σ operator

$$\begin{aligned}\Sigma_N&=\left\langle N\left|\int d\mathbf{x}\Sigma(x)\right|N\right\rangle_{\text{connected}}\\&=\left\langle\int d\mathbf{x}\Sigma(x)\right\rangle_N-\left\langle\int d\mathbf{x}\Sigma(x)\right\rangle_N^{\text{disconnected}}\\&=\frac{1}{3}\sum_{a=b=1}^{3}\langle N|[Q_5^a,[Q_5^b,H_{\chi\mathrm{SB}}]]|N\rangle\\&\quad-(2\pi)^3\delta^{(3)}(0)\frac{1}{3}\sum_{a=b=1}^{3}\langle 0|[Q_5^a,[Q_5^b,\mathcal{H}_{\chi\mathrm{SB}}(0)]]|0\rangle\\&=\int d\mathbf{x}\{\langle\Sigma(x)\rangle_N-\langle\Sigma(0)\rangle_0\},\end{aligned}\tag{40}$$

where $H_{\chi\mathrm{SB}}=\int d\mathbf{x}\mathcal{H}_{\chi\mathrm{SB}}(x)$. In this application it is preferable to quantize the system in a finite volume Ω, so as to avoid dealing with a new infinity in the form of a Dirac delta function of zero argument, $(2\pi)^3\delta^{(3)}(0)=\lim_{\Omega\to\infty}(\Omega=\int_\Omega d\mathbf{x})$. Subtraction of the disconnected term proceeds naturally using the equations of motion.

Initially we have

[1]Note that two sets of χSB terms may effectively mirror each other under a ''lower'' chiral symmetry like $\mathrm{SU}_L(2)\times\mathrm{SU}_R(2)$, but be very different under a ''higher'' symmetry such as $\mathrm{SU}_L(3)\times\mathrm{SU}_R(3)$. For example, the chiral transformation properties of both the current quark $\mathcal{H}_{q\chi\mathrm{SB}}$ and the bare nucleon mass term $\mathcal{H}_{N\chi\mathrm{SB}}$ are those of $(2,\bar{2})\oplus(\bar{2},2)$. However, in the $N_f=3$ case the quarks form an SU(3) triplet, which means that their bare mass terms transform as $(3,\bar{3})\oplus(\bar{3},3)$, whereas the spin-1/2 baryons are part of an SU(3) octet, which means that their χSB terms transform as either (8,8) or $(8,1)\oplus(1,8)$ under the chiral $\mathrm{SU}_L(3)\times\mathrm{SU}_R(3)$ group [14]. This group theoretical difference implies different pseudoscalar meson mass spectra in these two models of χSB. Since we know that the observed pseudoscalar masses conform rather well with the current quark mass model [14], we are forced to conclude that the baryon-antibaryon contribution to the pseudoscalar mass spectrum is supressed. This raises the question to what extent one may apply the baryon current mass model of χSB and $\varepsilon_3\neq 0$ in the two-flavor sector.

[2]This accounts for the different dimensions of the vacuum and nucleon Σ terms.

$$
\begin{aligned}
\frac{1}{3}\sum_{a=b=1}^{3}\int d\mathbf{x}[Q_5^a,[Q_5^b,\mathcal{H}_{\chi\mathrm{SB}}(x)]]
&=\int d\mathbf{x}\left[-\varepsilon_1\sigma-2\varepsilon_2\left(\sigma^2-\frac{1}{3}\boldsymbol{\pi}^2\right)+\varepsilon_3\bar{\psi}\psi\right]\\
&=-\Omega(\varepsilon_1+2\varepsilon_2 f_\pi)f_\pi-\int d\mathbf{x}[s(\varepsilon_1+4\varepsilon_2 f_\pi)-\varepsilon_3\bar{\psi}\psi]\\
&\quad+\mathcal{O}(s^2)+\mathcal{O}(\boldsymbol{\pi}^2).
\end{aligned}
\tag{41}
$$

Using Eq. (5) we obtain the equations of motion for the shifted σ field s

$$
\begin{aligned}
[\partial^2+m_\sigma^2]s=&-g_0\bar{\psi}\psi-\left(\frac{m_\sigma^2-m_\pi^2+2\varepsilon_2}{2f_\pi}\right)(3s^2+\boldsymbol{\pi}^2)\\
&+\left(\frac{m_\sigma^2-m_\pi^2+2\varepsilon_2}{2f_\pi^2}\right)s(s^2+\boldsymbol{\pi}^2)+\left(\frac{g_A-1}{2f_\pi^2}\right)\\
&\times[\partial^\mu(\bar{\psi}\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}\psi)+(\bar{\psi}\gamma_\mu\gamma_5\boldsymbol{\tau}\psi)\cdot(\partial^\mu\boldsymbol{\pi})].
\end{aligned}
\tag{42}
$$

The lowest-order perturbative solution is the following integral equation defined in terms of the free Klein-Gordon Green's function $\Delta_F(x;m_\sigma)$:

$$
\begin{aligned}
s(x)=\int d^4y\,\Delta_F(x-y;m_\sigma)\bigg\{&g_0\bar{\psi}(y)\psi(y)\\
&+\left(\frac{m_\sigma^2-m_\pi^2+2\varepsilon_2}{2f_\pi}\right)[3s^2(y)+\boldsymbol{\pi}^2(y)]\\
&+\left(\frac{m_\sigma^2-m_\pi^2+2\varepsilon_2}{2f_\pi^2}\right)s(y)[s^2(y)+\boldsymbol{\pi}^2(y)]+\left(\frac{g_A-1}{2f_\pi^2}\right)\\
&\times[\partial^\mu(\bar{\psi}\gamma_\mu\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}\psi)+(\bar{\psi}\gamma_\mu\gamma_5\boldsymbol{\tau}\psi)\cdot(\partial^\mu\boldsymbol{\pi})]\bigg\},
\end{aligned}
\tag{43}
$$

which upon inserting into the definition (41) leads to

$$
\begin{aligned}
\Sigma_N&=\langle\Sigma\rangle_N^{\mathrm{connected}}\\
&=\frac{1}{3}\sum_{a=b=1}^{3}\int d\mathbf{x}\langle N|[Q_5^a,[Q_5^b,\mathcal{H}_{\chi\mathrm{SB}}(x)]]|N\rangle_{\mathrm{connected}}\\
&=-(\varepsilon_1+4\varepsilon_2 f_\pi)\int d\mathbf{x}\langle s(x)\rangle_N-\varepsilon_3\int d\mathbf{x}\langle\bar{\psi}(x)\psi(x)\rangle_N\\
&\quad+\mathcal{O}(s^2)+\mathcal{O}(\boldsymbol{\pi}^2)\\
&=-g_0(\varepsilon_1+4\varepsilon_2 f_\pi)\int d\mathbf{x}\int d^4y\,\Delta_F(x-y;m_s)\\
&\quad\times\langle\bar{\psi}(y)\psi(y)\rangle_N+\varepsilon_3\int d\mathbf{x}\langle\bar{\psi}(x)\psi(x)\rangle_N\\
&\quad+\mathcal{O}(s^2)+\mathcal{O}(\boldsymbol{\pi}^2)\\
&=\frac{g_0}{m_\sigma^2}(\varepsilon_1+4\varepsilon_2 f_\pi)+\varepsilon_3+\mathcal{O}(\varepsilon^2)\cdots,
\end{aligned}
\tag{44}
$$

where the dots represents higher order terms of the fields which are neglected since we are working within the tree approximation. Using Eq. (6a) and the tree approximation result, Eq. (9), we find

$$
\Sigma_N=M\left(\frac{m_\pi^2+2\varepsilon_2}{m_\sigma^2}\right)+\varepsilon_3+\mathcal{O}(\varepsilon^2).
\tag{45}
$$

Naively we expect the value of Σ_N to be given by the sum of the current quark masses [11] $\propto\varepsilon_3$ which is reflected in a nonzero ''bare'' nucleon mass. The presence of the scalar field which induces the spontaneous chiral symmetry breaking in our model, changes radically the value of Σ_N, see Eq. (45). We shall return to this expression in Sec. IV C. Since there are no elementary scalar fields in the QCD Lagrangian it is difficult to demonstrate how this could happen in QCD but we note that there are scalar *bound* states in QCD.[3]

B. The Σ term from the scattering amplitude

The t-dependent pion nucleon Σ_N term can also be defined in terms of the on-mass-shell isospin symmetric amplitude as follows [6,12,13]:

$$
\begin{aligned}
D^{(+)}(\nu,\nu_B,k_1^2=m_\pi^2,\ k_2^2=m_\pi^2)
&\equiv D^{(+)}_{\mathrm{PV\,Born}}(\nu,\nu_B,\ k_1^2=m_\pi^2,\ k_2^2=m_\pi^2)\\
&\quad+\frac{\Sigma_N(t)}{f_\pi^2},
\end{aligned}
\tag{46a}
$$

$$
D^{(+)}_{\mathrm{PV\,Born}}(\nu,\nu_B,k_1^2=m_\pi^2,\ k_2^2=m_\pi^2)=\left(\frac{g^2_{\pi N}}{M}\right)\frac{\nu_B^2}{\nu_B^2-\nu^2},
\tag{46b}
$$

where we have defined $D^{(+)}_{\mathrm{PV\,Born}}$ as given by the diagrams, Figs. 1(a) and 1(b), using a pure pseudovector (PV) πN interaction Lagrangian. Equivalently

$$
\tilde{D}^{(+)}(\nu=0,\ \nu_B=0,\ k_1^2=m_\pi^2,\ k_2^2=m_\pi^2)=\frac{\Sigma_N(t=2m_\pi^2)}{f_\pi^2},
\tag{47}
$$

which when evaluated at the unphysical Cheng-Dashen point gives the value of $\Sigma_N(t=2m_\pi^2)$. Here as usual

$$
\tilde{D}^{(\pm)}=D^{(\pm)}-D^{(\pm)}_{\mathrm{PV\,Born}}.
\tag{48}
$$

When we compare Eq. (47) with Eq. (23), we obtain the expression

[3] As a simple illustration of this point one may take the example of the NJL model in which there are no elementary scalar mesons, but the fermion (in that case the constituent quark) Σ term is dominated by the scalar bound state's contribution.

$$\Sigma_N(t)=\varepsilon_3-M\left(\frac{m_\pi^2-t-2\varepsilon_2}{m_\sigma^2-t}\right)+\mathcal{O}(\varepsilon^2),$$

$$\Sigma_N(t=2m_\pi^2)\simeq\varepsilon_3+M\left(\frac{m_\pi^2+2\varepsilon_2}{m_\sigma^2}\right)+\mathcal{O}(\varepsilon^2), \qquad (49)$$

where in the last step we assume $m_\pi^2 \ll m_\sigma^2$, and as above we assume $m_\pi^2 \propto \varepsilon_i$, $i=1,2,3$. Equation (49) is in agreement with the canonical result of Eq. (45) and with the original linear Σ model result of Campbell [6].

C. Comparison with experiment

The Σ operator, Eq. (29), is often identified with the chiral symmetry breaking Hamiltonian itself. In two of the three cases in Eq. (33), the nucleon Σ term is a measure of the χSB in the nucleon. In those cases it equals the shift of the nucleon mass δM due to the χSB terms in the Hamiltonian. This reasoning underlies the standard interpretation of the nucleon Σ term as being a measure of the strangeness content of the nucleon [11]. A large value of $\Sigma_N \simeq 65$ MeV has often been interpreted as a sign of a substantial $s\bar{s}$ content of the nucleon. We shall show that in the extended linear Σ model, Eq. (2), such a large values for $\Sigma_N(t=2m_\pi^2)$ can be obtained without any strangeness content of the nucleon.

In the tree approximation the value of the Σ_N term in terms of the values of the parameters α_i of Eqs. (39a)–(39c) is

$$\Sigma_N=M_N^0(1-\alpha_1-\alpha_2)+M(1+\alpha_2)\left(\frac{m_\pi^2}{m_\sigma^2}\right)+\mathcal{O}(\varepsilon^2), \qquad (50)$$

where we use $\overline{M}_N^0 \simeq 23$ MeV, $M=940$ MeV, and $m_\pi=140$ MeV. For possible values of the σ-meson mass in the interval $m_\sigma=400$–1400 MeV [2] we have $M(m_\pi/m_\sigma)^2=115$–9 MeV, and hence

$$\Sigma_N=(1-\alpha_1-\alpha_2)\times 23\ \text{MeV}+(1+\alpha_2)\times(115-9)\ \text{MeV}+\mathcal{O}(\varepsilon^2). \qquad (51)$$

This range of values easily encompasses the experimentally allowed range of 45–75 MeV, for sufficiently light m_σ and for reasonable values of ε_i, $i=1,2,3$. Note, however, that due to the large uncertainty in m_σ this experimental value can not be used to effectively fix the above linear combination of the α_i parameters.

To compare the πN scattering lengths, Eqs. (27) and (28), with experimental values, we discuss the general case $\varepsilon_i \neq 0$, $i=1,2,3$:

$$a_0^{(+)}\simeq\frac{-g_{\pi N}^2}{4\pi(1+m_\pi/M)}\left(\frac{m_\pi}{M}\right)\times\left\{\left(\frac{m_\pi}{2M}\right)^2+g_A^{-2}\left[(1-\alpha_2)\left(\frac{m_\pi^2}{m_\sigma^2}\right)-\alpha_3\left(\frac{M_N^0}{M}\right)\right]\right\}\times\frac{1}{m_\pi}+\mathcal{O}(\varepsilon^2) \qquad (52)$$

and

$$a_0^{(-)}\simeq\frac{g_{\pi N}^2}{8\pi(1+m_\pi/M)}\left(\frac{m_\pi}{M}\right)^2\frac{1}{g_A^2}\frac{1}{m_\pi}+\mathcal{O}(\varepsilon_3^2). \qquad (53)$$

Despite the tiny "bare" nucleon mass $M_N^0 \ll M$ the value of the isoscalar scattering length $a_0^{(+)}$ shows significant dependence on both the α_2 and α_3 parameters for values of $m_\sigma \leq M_N$. In the extended linear Σ model the theoretical value of $a_0^{(+)}$ can easily reproduce the "old" experimental value $a_0^{(+)}|_{\text{expt}}=-0.010(4)m_\pi^{-1}$, Ref. [16], and can have either sign with extreme values of α_i parameters. Recent pionic atom experiments allow for $a_0^{(+)}$ values of comparable size of either sign if only hydrogen data are taken into account [17]. The addition of the latest pionic deuteron data can flip the sign and definitely reduces both the mean value and the uncertainties [17]. The new experimental value for $a_0^{(+)}|_{\text{expt}} \simeq \pm 0.0020(16)m_\pi^{-1}$ is much (almost 50 times) smaller than the "natural" size obtained from the usual $\mathcal{L}_{\chi\text{SB}}$ and requires further cancellations among these small terms. Thus, this latest value of $a_0^{(+)}|_{\text{expt}}$ appears to be of $\mathcal{O}(\varepsilon^2)$. In order that our $\mathcal{O}(\varepsilon^2)$ calculation of the $a_0^{(+)}$ S-wave scattering length (52) reproduce this very small experimental value, a very delicate cancellation between the various terms must take place in our model that makes it very sensitive to both α_2 and α_3 and to the value of m_σ. We conclude that the present approximate calculation is not sufficiently precise to be reliably and profitably compared with the most recent data.

To $\mathcal{O}(\varepsilon)$ the isospin antisymmetric scattering wave scattering length $a_0^{(-)}$ is independent of α_i. The leading order (Weinberg-Tomozawa) prediction (53) is within one standard deviation from the (old) mean experimental value $a_0^{(-)}|_{\text{expt}}=0.091(2)m_\pi^{-1}$, Ref. [16]. The new experimental value of $a_0^{(-)}|_{\text{expt}}=0.0868(14)m_\pi^{-1}$, Ref. [17], is subject to the same caveats as for the isoscalar one described above.

V. RELATIONSHIP TO CHIRAL PERTURBATION THEORY

A "chiral rotation" defined in the limit $m_\sigma \to \infty$ by

$$N=\sqrt{\mathcal{R}}(1+i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\xi})\psi, \qquad (54a)$$

$$\boldsymbol{\pi}=\mathcal{R}\boldsymbol{\phi}, \qquad (54b)$$

$$\sigma=f_\pi\mathcal{R}(1-\boldsymbol{\xi}^2), \qquad (54c)$$

$$\mathcal{R}=\left[1+\left(\frac{\boldsymbol{\phi}}{2f_\pi}\right)^2\right]^{-1}=[1+\boldsymbol{\xi}^2]^{-1}, \tag{54d}$$

leads, by way of standard arguments [18,19] from the linear Σ model Lagrangian without the extra derivative interaction terms in Eq. (2), to the nonlinear one

$$\mathcal{L}-\mathcal{L}_{\chi\mathrm{SB}}=\bar{N}[i\not\partial-M]N+\frac{1}{2}\mathcal{R}^2(\partial_\mu\boldsymbol{\phi})^2+\mathcal{R}\left(\frac{1}{2f_\pi}\right)\times(\bar{N}\gamma_\mu\gamma_5\boldsymbol{\tau}N)\cdot\partial^\mu\boldsymbol{\phi}-\mathcal{R}\left(\frac{1}{2f_\pi}\right)^2\times(\bar{N}\gamma_\mu\boldsymbol{\tau}N)\cdot(\boldsymbol{\phi}\times\partial^\mu\boldsymbol{\phi}). \tag{55}$$

The above form of the nonlinear Lagrangian (55) differs from Weinberg's [18] by the absence of an *ad hoc* factor g_A in front of the ''pseudovector'' coupling term. The source of this difference, as emphasized by Weinberg himself, was the need to have both the empirical g_A factor in the axial current and the correct two-pion-nucleon contact interaction. We shall now show that the extended linear Σ model, Eq. (2), leads to Weinberg's nonlinear Σ model Lagrangian, i.e., that the extra terms in Eq. (2) promoted by Bjorken-Nauenberg and by Lee provide precisely the difference prescribed *ad hoc* by Weinberg. The extra terms in Eq. (2) can be written in terms of the currents $\mathbf{V}_\mu,\mathbf{v}_\mu$ and $\mathbf{A}_\mu,\mathbf{a}_\mu$:

$$\begin{aligned}\mathcal{L}_{\mathrm{bn}}&=\left(\frac{g_A-1}{f_\pi^2}\right)\left[\left(\bar{\psi}\gamma_\mu\frac{\boldsymbol{\tau}}{2}\psi\right)\cdot(\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi})+\left(\bar{\psi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\psi\right)\cdot(\sigma\partial^\mu\boldsymbol{\pi}-\boldsymbol{\pi}\partial^\mu\sigma)\right]\\&=\left(\frac{g_A-1}{f_\pi^2}\right)[\mathbf{V}_\mu\cdot\mathbf{v}^\mu+\mathbf{A}_\mu\cdot\mathbf{a}^\mu].\end{aligned} \tag{56}$$

The $\mathbf{V}_\mu^a$ and $\mathbf{v}_\mu^a$ are

$$\mathbf{V}_\mu^a=\mathcal{R}\left\{(1-\boldsymbol{\xi}^2)\left(\bar{N}\gamma_\mu\frac{\boldsymbol{\tau}}{2}N\right)^a-\bar{N}\gamma_\mu\gamma_5(\boldsymbol{\tau}\times\boldsymbol{\xi})^aN+\boldsymbol{\xi}^a\bar{N}\gamma_\mu(\boldsymbol{\tau}\cdot\boldsymbol{\xi})N\right\}, \tag{57}$$

$$\mathbf{v}_\mu^a=\mathcal{R}^2(\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\phi})^a. \tag{58}$$

Similarly,

$$\mathbf{A}_\mu^a=\mathcal{R}\left\{(1-\boldsymbol{\xi}^2)\left(\bar{N}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}N\right)^a-\bar{N}\gamma_\mu(\boldsymbol{\tau}\times\boldsymbol{\xi})^aN+\boldsymbol{\xi}^a\bar{N}\gamma_\mu\gamma_5(\boldsymbol{\tau}\cdot\boldsymbol{\xi})N\right\}, \tag{59}$$

$$\mathbf{a}_\mu^a=\mathcal{R}^2f_\pi[\partial^\mu\boldsymbol{\phi}^a(1-\boldsymbol{\xi}^2)+2\boldsymbol{\xi}^a(\boldsymbol{\xi}\cdot\partial^\mu\boldsymbol{\phi})]. \tag{60}$$

Inserting these into Eq. (56) we find

$$\mathcal{L}_{\mathrm{bn}}=(g_A-1)\mathcal{R}(\bar{N}\gamma_\mu\gamma_5\boldsymbol{\tau}N)\cdot\partial^\mu\boldsymbol{\xi}, \tag{61}$$

which when combined with Eq. (55) leads to Weinberg's nonlinear Lagrangian with $g_A\neq 1$. One can now write the resulting nonlinear Lagrangian in the notation of chiral perturbation theory and thus convince oneself that this is equivalent to the lowest order Lagrangian of Gasser, Sainio, and Švarc (GSŠ) [20,21]. Conversely, one should be able to convert finite-chiral-order terms in the GSŠ nonlinear chiral Lagrangian into extended linear ones. This is more than an academic point, for it makes it clear that the choice between the linear and nonlinear realizations is a matter of convenience. Quite often it is more expedient to work in the representation wherein one has the σ, or s fields from the beginning, rather than building it up from the pions. Moreover, the linear Lagrangian is always a polynomial in the meson fields, rather than a fractional, or even (transcedental) exponential function of π as is the case in the nonlinear realization.

VI. SUMMARY AND CONCLUSIONS

In this work we have shown that the extension of the linear Σ model allows $g_A\neq 1$ in the axial current in the linear realization of chiral symmetry. The chiral charge algebra holds in the extended linear Σ model despite the fact that the spatial part of the nucleon axial current is renormalized by g_A, because the nucleon axial charge is not renormalized.

We evaluated the elastic πN scattering amplitude in the tree approximation with three kinds of χSB terms similar to Ref. [6]. The $a_0^{(-)}$ scattering length is now in agreement with the Weinberg-Tomozawa result, and we can obtain a very small $a_0^{(+)}$ scattering length value in contrast to the original linear Σ model.

The Σ_N term with three different χSB terms was also evaluated. In the tree approximation the Σ_N term from the canonical operator definition using the equations of motion coincides with the result derived from the πN scattering amplitude. The vacuum matrix element of the Σ operator puts one constraint on a linear combination of the three different χSB parameters ε_i, $i=1,2,3$. It is noteworthy that in our extended linear Σ model a large value for Σ_N can easily be obtained without any $s\bar{s}$ components in the nucleon. The reason for this is that the scalar σ meson can make a large contribution to Σ_N depending on the value of the mass m_σ. Finally we showed that a chiral rotation of the extended linear Σ model Lagrangian leads to the lowest-order πN χPT Lagrangian in the limit $m_\sigma\to\infty$.

We close with several suggestions for future research: (i) Derive ε_i, for $i=1,2,3$ from quark models or QCD (for a sketch of such a derivation in the NJL model, see Appendix B). (ii) Apply the extended Σ model to a re-evaluation of a possible pion condensation in nuclear matter, where $g_A\neq 1$ is very important, but has not been consistently implemented to date. (iii) Establish a relation between the free parameters of the extended linear Σ model and the low-energy constants in the χPT Lagrangian.

Note added in proof. After this paper was accepted it was brought to our attention that a paper by Carter, Ellis, and Rudaz [24] covers some of the same ground.

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APPENDIX A: DASHEN'S RELATION AT THE ONE-LOOP LEVEL

Here we follow Sec. V of Ref. [4] and show that at the one-nucleon-loop level we can derive Eq. (34). (Analogous calculations at the one-meson-loop self-consistent approximation level can be performed along the lines of Ref. [22].)

The Hartree+RPA approximation can be defined by three Schwinger-Dyson integral equations: (i) the zero-body or vacuum equation, (ii) the one-body or the fermion mass gap equation, and (iii) the two-body or one-meson Bethe-Salpeter equation, shown in Figs. 5(a), 5(b), and 5(c) of Ref. [4], respectively. The Bethe-Salpeter equation for the $N\bar{N}$ pseudoscalar scattering amplitudes is separable and has as an exact solution in Hartree+RPA the following expression:

$$D_{\pi}(k)=\frac{1}{k^2-\Sigma_{\pi}^{(\mathrm{RPA})}(k)}, \tag{A1}$$

where $\Sigma^{(\mathrm{RPA})}(k)$ is a sum of a single one-nucleon-loop polarization diagram plus one ''tree'' diagram. The Schwinger-Dyson equations now read ($v=f_{\pi}$) [4]

$$v=-\frac{\varepsilon_1}{\mu_0^2}+\lambda_0\frac{v^3}{\mu_0^2}+\frac{i}{\mu_0^2}g_0N_f\int\frac{d^4p}{(2\pi)^4}\frac{4M}{p^2-M^2}, \tag{A2a}$$

$$M=M_0+g_0v=\varepsilon_3+g_0v, \tag{A2b}$$

$$\Sigma_{\pi}^{(\mathrm{RPA})}(k)=2\varepsilon_2-\mu_0^2+\lambda_0v^2+g_0^2\Pi_{\pi}^{(\mathrm{RPA})}(k), \tag{A2c}$$

where Eq. (A2b) is the same as Eq. (6a). The pion polarization function $\Pi_{\pi}^{(\mathrm{RPA})}(k)$ can be written as

$$\begin{aligned}\Pi_{\pi}^{(\mathrm{RPA})}(k)&=4iN_f\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2-M^2}-2iN_fk^2I(k)\\&=\frac{1}{M}\langle\bar{\psi}\psi\rangle_0-2iN_fk^2I(k),\end{aligned} \tag{A3}$$

where we introduced the logarithmically divergent integral

$$I(k)=\int\frac{d^4p}{(2\pi)^4}\frac{1}{[p^2-M^2][(p+k)^2-M^2]}. \tag{A4}$$

In order to prove the Dashen relation (34) we rewrite Eq. (A2a) using Eq. (A3) as follows

$$-\mu_0^2+\lambda_0v^2=\frac{\varepsilon_1}{v}-\frac{g_0}{v}\langle\bar{\psi}\psi\rangle_0. \tag{A5}$$

When we compare this equation with the tree approximation results, Eq. (6c), we see that the last term above is beyond the tree-level. Insert this into Eq. (A2c) to find to lowest order in ε_i (as $k\to0$):

$$\begin{aligned}m_{\pi}^2=\Sigma_{\pi}^{(\mathrm{RPA})}(0)&=2\varepsilon_2-\mu_0^2+\lambda_0v^2+\frac{g_0^2}{M}\langle\bar{\psi}\psi\rangle_0\\&=\frac{\varepsilon_1}{v}-\frac{g_0}{v}\langle\bar{\psi}\psi\rangle_0+2\varepsilon_2+\frac{g_0^2}{M}\langle\bar{\psi}\psi\rangle_0\\&\simeq\frac{\varepsilon_1}{v}+2\varepsilon_2-\frac{\varepsilon_3}{v^2}\langle\bar{\psi}\psi\rangle_0,\end{aligned} \tag{A6}$$

where we used the GT relation (A2b). Equation (A6) is equivalent to Eq. (34) to leading order in χSB parameters. Thus we have demonstrated the necessity of a self-consistent gap equation for the validity of Dashen's formula when χSB is determined by Eq. (3).

APPENDIX B: SKETCH OF A DERIVATION OF ε_1 AND ε_3

We shall use the bosonization technique in a simple chiral quark (NJL) model to show that ε_1 is related to ε_3 at the quark level. This is just a sketch meant to illustrate an approach to the more challenging case of nucleons.

The NJL model Lagrangian density is

$$\mathcal{L}_{\mathrm{NJL}}=\bar{\psi}[i\partial\!\!\!/-m^0]\psi+G[(\bar{\psi}\psi)^2+(\bar{\psi}i\gamma_5\boldsymbol{\tau}\psi)^2]. \tag{B1}$$

The substitution

$$-g_0\sigma=G(\bar{\psi}\psi), \tag{B2a}$$

$$-g_0\boldsymbol{\pi}=G(\bar{\psi}i\gamma_5\boldsymbol{\tau}\psi), \tag{B2b}$$

for one of the two quark bilinears leads to the (semi-bosonized) linear σ model interaction Lagrangian

$$\mathcal{L}_{\mathrm{int}}=-g_0\bar{\psi}[\sigma+i\gamma_5\boldsymbol{\pi}\cdot\boldsymbol{\tau}]\psi. \tag{B3}$$

The chiral symmetry breaking current quark mass term is

$$\mathcal{L}_{\chi\mathrm{SB}}=-m^0\bar{\psi}\psi=-\varepsilon_3\bar{\psi}\psi=m^0\frac{g_0}{G}\sigma=\varepsilon_1\sigma. \tag{B4}$$

Note that Eq. (B2a) implies (using the linear Σ model relations)

$$-g_0\langle\sigma\rangle_0=G\langle\bar{\psi}\psi\rangle_0=-g_0f_{\pi}=-m. \tag{B5}$$

This in turn leads to

$$\varepsilon_1=\varepsilon_3\frac{g_0}{G}=-\frac{m^0}{f_{\pi}}\langle\bar{\psi}\psi\rangle_0=m_{\pi}^2f_{\pi}, \tag{B6}$$

where the last step follows from the GMOR relation, which can be explicitly demonstrated in the NJL model at the quark level.

Chiral symmetry breaking coefficients have been calculated at the mesonic level in a more sophisticated chiral quark (''global color'') model in Ref. [15]. The challenge is to extend this analysis to the nucleon case. This can presumably be done by solving the three-quark Faddeev-Bethe-Salpeter equation, see Ref. [23], and calculate ε_3 at the nucleon level.

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